

AN INTEGER PROGRAMMING APPROACH TO EQUITABLE COLORING PROBLEMS

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Resumo

Uma coloração equilibrada é uma coloração própria de um Grafo G na qual, para quaisquer duas cores, a diferença entre as ocorrências das cores é no máximo igual a uma unidade. Para muitas aplicações faz-se necessário resolver o problema associado de encontrar a coloração equilibrada com o menor número de cores. O conceito de coloração equilibrada e o problema de minimização associado podem ser estendidos para coloração de arestas e coloração total. Neste trabalho introduz-se uma formulação por programação inteira para o problema de coloração equilibrada, e também as transformações necessárias para se resolver os problemas correlatos de coloração de arestas e coloração total.

Palavras-chave: Coloração Equilibrada, Programação Inteira.

Abstract

An equitable coloring is a proper coloring of a graph G in which, for any two colors, the difference between color occurrence is at most one. For many applications it is necessary to solve the associated problem of finding the equitable coloring with the minimum number of colors. The concept of equitable coloring and the associated problem can be extended to edge and total coloring. In this work we introduce an integer programming formulation to the equitable coloring problem, as well as the necessary transformations to solve the correlated equitable edge coloring and equitable total coloring problems.

Keywords: Equitable Coloring, Integer Programming.

1. INTRODUCTION

The concept of equitable coloring has many applications, for example in scheduling problems, when we need to distribute tasks among agents so that any two agents must have similar quantity of tasks.

Equitable coloring can be used also to solve load balancing problems, when we need to distribute process and communications activities evenly across a computer network so that no single device is overwhelmed. Load balancing is specially important for networks where it is difficult to predict the number of requests that will be issued to a server. Load balancing can also refer to the communications channels themselves, in this case we can use equitable edge coloring or equitable total coloring, depending if we need or not to distribute the tasks only among the channels or consider the servers too.

Recently we have used balanced total coloring to create general algorithms for interconnecting networks. Total coloring of a graph can suitable represent interconnecting networks, since the elements that can work at the same time are mapped to the classes of colors representing independents sets. This allows the creation of algorithms based on colors and not in topologies themselves. We can enumerate much more applications, but the precedent list is sufficient to show the importance of the concept and the necessity of finding the solution of the associate problem, i.e, find the equitable coloring of a graph with the minimum number of colors, in a reasonable computational time.

In the present work we introduce an integer programming formulation to the equitable coloring problem, as well as the necessary transformations to solve the correlated equitable edge coloring and equitable total coloring problems. It is important to emphasize that, as far as we know, there is no another mathematical programming formulation for the problem in the literature.

This work is organized as follows. The second section is dedicated to formal definitions of equitable coloring problems, and the necessary transformations to convert the equitable edge coloring and the equitable total coloring problems into the equitable coloring problem. The third section is dedicated to present the integer programming formulation and in the last section we show some preliminary computational results.

2. DEFINITIONS

A *proper coloring* φ of a graph $G(V, E)$ is a mapping from V to a set of colors C , such that no two adjacent vertices have the same image.

A *proper edge coloring* φ' of a graph $G(V, E)$ is a mapping from E to a set of colors C , such that no two adjacent edges have the same image.

A *proper total coloring* φ^T of a graph $G(V, E)$ is a mapping from $V \cup E$ to a set of colors C , such that no two adjacent or incident elements have the same image.

Given a proper coloring of a graph G , we denote by $y(c)$ the number of times that color c appears in the coloring.

A proper coloring of G is an *equitable coloring* if, for every two (node) colors c_1 and c_2 of C , we have $|y(c_1) - y(c_2)| \leq 1$. The associated *equitable coloring problem* consists in finding the equitable coloring of G with the minimum number of colors. This number is denoted by $\chi_{eq}(G)$.

A proper edge coloring of G is an *equitable edge coloring* if, for every two (edge) colors c_1 and c_2 of C , we have $|y(c_1) - y(c_2)| \leq 1$. The associated *equitable edge coloring problem* consists in finding the equitable edge coloring of G with the minimum number of colors. This number is denoted by $\chi'_{eq}(G)$.

A proper total coloring of G is an *equitable total coloring* if, for every two (node or edge) colors c_1 and c_2 of C , we have $|y(c_1) - y(c_2)| \leq 1$. The associated *equitable total coloring problem* consists in finding the equitable total coloring of G with the minimum number of colors. This number is denoted by $\chi_{eq}^T(G)$.

The three problems are refereed as *equitable coloring problems*.

Given a graph $G(V, E)$, the *line graph* of G is a graph $\tilde{G}(\tilde{V}, \tilde{E})$ in which $\tilde{V} = E$ and $xw \in \tilde{E}$ if and only if x and w has a common vertex in G .

Given a graph $G(V, E)$ the *total graph* of G is a graph $\dot{G}(\dot{V}, \dot{E})$ in which $\dot{V} = V \cup E$ and $xw \in \dot{E}$ if and only if one of the following sentences is true:

- (1) x and w are edges of G sharing a common vertex.
- (2) x and w are adjacent vertices in G .
- (3) $x \in V$ and $w \in E$ and w is incident to x in G .

It is easy to see that an equitable coloring of the line graph of a graph G is equivalent to an equitable edge coloring of G . It is also easy to see that an equitable coloring of the total graph of a graph G is equivalent to an equitable total coloring of G . Therefore, the integer programming formulation introduced in next section can be used to model all the equitable coloring problems.

3. INTEGER PROGRAMMING FORMULATION

In this section we introduce the Integer Programming Formulation to The Equitable Coloring Problem. It is important to emphasize that, as far as we know, there is no another mathematical programming formulation for this problem in the literature.

The first step consists in introducing the decision variables, all integer. There are three classes of variables: the binary x_{vc} variables linking colors c to vertices v , the also binary w_c variables related to colors c and the integer variables y_c related also to colors c .

The first class of binary variables, x_{vc} , is widely used in the literature for coloring problems, and are defined as follows:

$$(1) \quad x_{vc} = \begin{cases} 1 & \text{if color } c \text{ is assigned to vertex } v, \\ 0 & \text{otherwise,} \end{cases}$$

for all $v \in V$ and $c \in \{1, \dots, \Delta + 1\}$.

It is known [Hajnal-Szemerédi] that every graph G has $\chi_{eq}(G) \leq \Delta(G) + 1$. Then $\Delta + 1$ colors are sufficient to color a graph in an equitable way. So, in order to minimize the number of colors being used in the formulation, we define the class w_c of binary variables as follows:

$$(2) \quad w_c = \begin{cases} 1 & \text{if } x_{vc} = 1 \text{ for some vertex } v, \\ 0 & \text{otherwise,} \end{cases}$$

for all $c \in \{1, \dots, \Delta + 1\}$.

Until this point we have not address the equitable necessary condition stating that an equitable coloring of a graph G is a proper vertex coloring such that the sizes of every two used color classes differ by at most one unity. Here the size of a used color class is the number of vertices colored with that color. In order to address this question we introduce the class of y_c integer variables as follows:

$$(3) \quad y_c = \sum_{v=1}^n x_{vc}, \text{ for all } c \in \{1, \dots, \Delta + 1\}.$$

The mathematical formulation is then stated as follows:

$$(4) \quad \left\{ \begin{array}{ll} \min \sum_{c=1}^{\Delta+1} w_c & \\ \sum_{c=1}^{\Delta+1} x_{vc} = 1 & , \forall v \in V \quad (a) \\ x_{vc} + x_{uc} \leq w_c & , \forall (v, u) \in E, \forall c \in \{1, \dots, \Delta + 1\} \quad (b) \\ y_c = \sum_{v=1}^n x_{vc} & , \forall c \in \{1, \dots, \Delta + 1\} \quad (c) \\ |y_c - y_l| \leq 1 & , \forall c, l \in \{1, \dots, \Delta + 1\} \quad (d) \\ x_{vc} \in \{0, 1\} & , \forall v \in V, \forall c \in \{1, \dots, \Delta + 1\} \\ w_c \in \{0, 1\} & , \forall c \in \{1, \dots, \Delta + 1\} \\ y_c \text{ is integer} & , \forall c \in \{1, \dots, \Delta + 1\} \end{array} \right.$$

Constraints (4a) assert that each vertex must receive exactly one color and constraints (4b) say that no pair of adjacent vertices can share the same color. Variables y_c count the number of vertices colored with color c (constraints (4c)). Finally, constraints (4d) guarantee that the sizes of every two used color classes differ by at most one unity. Clearly, the variables w_c can be relaxed to $w_c \in [0, 1]$ without changing the feasible solution set.

In order to model (4) becomes implementable, we need to drop the modulus function in constraints (4d). It is easy to observe that, in fact, we have only to force $|y_c - y_l| \leq 1$ when both w_c and w_l are different from zero. This leads to the following pair of integer disjoint constraints that, together, replace (4d):

$$(5) \quad y_c - y_l \leq 1 + M(2 - w_c - w_l), \quad \forall c, l \in \{1, \dots, \Delta + 1\},$$

$$(6) \quad y_c - y_l \geq -1 - M(2 - w_c - w_l), \quad \forall c, l \in \{1, \dots, \Delta + 1\},$$

where M has to be big enough to force $|y_c - y_l| \leq 1$ only when both w_c and w_l are different from zero. Besides, it is also sufficient to choose among all pairs $c, l \in \{1, \dots, \Delta + 1\}$ that ones satisfying $c \neq l$ and $l > c$.

The new model obtained from (4) by replacing constraints (4d) with the pair of disjoint constraints (5) and (6) is implementable and has $(n + 2)(\Delta + 1)$ variables and $n + (m + 3)(\Delta + 1)$ constraints. The equitable coloring polytope \mathcal{ECP} is defined by the convex hull of all feasible solutions to this new integer problem.

4. COMPUTATIONAL RESULTS

In order to validate the model presented in Section 3 we have tested the integer programming formulation over some popular examples of the literature having a priori known equitable chromatic numbers.

The value for M was settled to $M = \min\{n, \Delta^2\}$ and the model was implemented using the commercial solver Xpress-MP 2006 with IVE 1.17.12, Mosel 1.6.3 and Optimizer 17.10.04. The computer used was a Pentium Intel Dual Core 2.66GHz, 960MB RAM with Windows XP Professional 2002.

Table 1 shows the obtained results in details. $G_{3,3}$ is a grid graph, S_7 is a star, W_6 is a wheel and the Kp,q graphs belong to the well known Kneser family of graphs. It is important to emphasize that we have found all the known $\chi_{eq}(G)$ for all graphs G tested. Therefore we have not shown gap column in this table.

G	n	m	Δ	$\chi_{eq}(G)$	Cardinality	CPU Time(s)
P_3	3	2	2	2	$\{1, 2\}$	0.01
P_5	5	4	2	2	$\{2, 3\}$	0.02
C_6	6	6	2	2	$\{3, 3\}$	0.01
C_7	7	7	2	3	$\{2, 3, 2\}$	0.08
K_3	3	3	2	3	$\{1, 1, 1\}$	0.03
K_5	5	10	4	5	$\{1, 1, 1, 1, 1\}$	0.16
$G_{3,3}$	9	12	4	2	$\{5, 4\}$	0.03
S_7	7	6	6	4	$\{2, 2, 2, 1\}$	0.40
W_6	6	10	5	4	$\{2, 1, 2, 1\}$	2.70
Cube	8	12	3	2	$\{4, 4\}$	0.03
Octahedron	6	12	4	3	$\{2, 2, 2\}$	0.05
Dodecahedron	20	30	3	3	$\{7, 7, 6\}$	0.09
Icosahedron	12	30	5	4	$\{3, 3, 3, 3\}$	71.25
$K_{5,2}$	10	15	3	3	$\{4, 3, 3\}$	0.05
$K_{7,3}$	35	70	4	3	$\{12, 11, 12\}$	0.70
$K_{9,4}$	126	315	10	3	$\{42, 42, 42\}$	1134.50

TABLE 1. Preliminary Computational Results

5. CONCLUSIONS AND FUTURE RESEARCH

This work presents an initial effort in order to explore and solve equitable coloring problems in an exact way. It is important to emphasize that, as far as we know, there was no another mathematical programming formulation for the problem in the literature.

Further experiments in heavier graphs (e.g. the DIMACS coloring benchmark graphs) will allow us to have a better appreciation of the real efficiency of our integer programming formulation.

Our future research will also address the characterization of the polytope \mathcal{ECP} trying to identify new families of valid inequalities and sufficient conditions for them to be facet-defining inequalities.

As a remark on the results in Kneser Graphs we can say they are somewhat surprising. In order to achieve the equitable coloration the Branch-and-Bound algorithm found coloring classes rather different from the canonical ones. This suggest a conjecture that, for Kneser graphs, $\chi_{eq} = \chi$ at least for the so called odd graphs, Kneser graphs of type $K(2n+1, n)$.

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