# THE $\bar{X}$ CHART'S PERFORMANCE IN THE PRESENCE OF THE MEASUREMENT ERROR AND THE AUTOCORRELATION 

Marcela A. G. Machado<br>Departamento de Produção, UNESP<br>Guaratinguetá, 12516-410, SP, Brasil<br>marcela@feg.unesp.br

Antonio F. B. Costa<br>Departamento de Produção, UNESP<br>Guaratinguetá, 12516-410, SP, Brasil<br>fbranco@feg.unesp.br<br>Philippe Castagliola<br>Universit'e de Nantes and IRCCyN<br>Nantes, France<br>philippe.castagliola@univ-nantes.fr


#### Abstract

Measurement error and the autocorrelation often exist in quality control applications. Both have an adverse effect on the $\bar{X}$ chart's performance. To counteract the undesired effect of the autocorrelation, we build up the samples with non-neighbor items, according to the time they were produced. To counteract the undesired effect of measurement error, we measure the quality characteristic of each item of the sample several times. The chart's performance is assessed when multiple measurements are applied and the samples are built by taking one item from the production line and skipping one, two or more before selecting the next.


KEYWORDS. Measurement error. Autocorrelation. $\overline{\boldsymbol{X}}$ chart's.


#### Abstract

RESUMO O erro de medição e a autocorrelação existem frequentemente em aplicações de controle de qualidade. Ambos têm um efeito adverso sobre o desempenho do gráfico de $\bar{X}$. Para neutralizar o efeito indesejado da autocorrelação, compõem-se as amostras com itens não vizinhos, considerando o momento em que foram produzidos. Para neutralizar o efeito indesejado do erro de medição, mede-se a característica de qualidade de cada item da amostra várias vezes. O desempenho do gráfico é avaliado quando múltiplas medidas são realizadas e as amostras são compostas tomando um item da linha de produção e pulando um, dois ou mais antes de escolher o próximo.


PALAVRAS CHAVE. Erro de medição. Autocorrelação. Gráfico de $\bar{X}$.

## 1. Introduction

Control charts are usually designed and evaluated assuming that the values of the quality characteristic $X$ are independent and free of measurement error. The literature on the statistical performance of control chart rarely contains any explicit mention of measurement error, see Linna and Woodall (2001). In recent papers, Shore (2004) studied the measurement error requirements to satisfy a pre-specified chart's performance, Linna et al. (2001a) studied the control of multivariate processes under measurement error and Maravelakis et al. (2004) studied the effect of measurement error on to EWMA charts. On the other hand, to deal with autocorrelated data various charting techniques have been proposed. If the interval between observations is short enough to produce correlation, then one simple approach is to skip some of them. The option of widening the control limits has also been suggested as a remedial method to deal with data autocorrelation, see Vasilopoulos and Stamboulis (1978). A more typical approach is to fit an appropriate time-series model to the observations (see Montgomery and Mastrangelo (1991) or Costa and Claro (2008)). The structure of the rest of the paper is as follow: in Section 2, we discuss the $\bar{X}$ chart's performance under measurement error. In section 3, we discuss the $\bar{X}$ chart's performance when the $X$ values adjust to a first-order autoregressive model and we also introduce the idea of building up the samples with non-neighbor items. In section 4, we assess the $\bar{X}$ chart's performance when the quality characteristic of each item of the sample is measured several times and the samples are built by taking one item from the production line and skipping one, two or more before selecting the next.

## 2. The Effect of the Measurement Error on the $\bar{X}$ Chart's Performance

In this paper the average run length (ARL) measures the efficiency of a control chart in detecting a process change. During the in-control period the $A R L=1 / \alpha$ and is called $A R L_{0}$, and during the out-of-control period the $A R L=1 / P$, being $P=1-\beta$ the power of the control chart. The risks $\alpha$ and $\beta$ are, respectively, the well-known Type I and Type II errors. A chart with a larger in-control $\operatorname{ARL}\left(A R L_{0}\right)$ indicates lower false alarm rate than other charts. A chart with a smaller out-of-control ARL indicates a better ability of detecting process shifts than other charts.

When measurement error is discussed in the literature (see Linna and Woodall (2001)), the usual measurement error model is

$$
\begin{equation*}
X=Y+\varepsilon_{i} \tag{1}
\end{equation*}
$$

where $Y$ is the true value of the quality characteristic, $\varepsilon$ is a random error term due to measurement imprecision, and $X$ is the observed result of some measurement operation. It is assumed that $X$ follows a normal distribution with mean $\mu_{0}$ and variance $\sigma^{2}(X)=\sigma_{m}^{2}+\sigma^{2}$, being $\sigma_{m}^{2}$ the variance of the measurement system and $\sigma^{2}$ the process variance. The expression of the sample mean is

$$
\bar{X}=\frac{m\left(Y_{1}+Y_{2}+\ldots Y_{n}\right)+\sum_{j=1}^{m} \sum_{i=1}^{n} e_{i j}}{n m}
$$

where $n$ is the size of the sample and $m$ is the number of times each item of the sample is measured. Consequently, the standard deviation of the sample means is

$$
\sigma(\bar{X})=\frac{\sqrt{\sigma^{2}+\frac{\sigma_{m}^{2}}{m}}}{\sqrt{n}}
$$

and the control limits of the $\bar{X}$ chart are

$$
C L=\mu_{0} \pm k \frac{1}{\sqrt{n}} \sqrt{\sigma^{2}+\sigma_{m}^{2} / m}
$$

where $k$ is a constant defined to satisfy some desired in-control $A R L=A R L_{0}$ (in this paper we adopted $k=3$, that is, $A R L_{0}=370.4$ ). If the mean of $Y$ shifts to $\mu_{1}=\mu_{0} \pm \delta \sigma$ the $\bar{X}$ chart has the following probability to signal

$$
P=\Phi(-k-C \delta \sqrt{n})+\Phi(-k+C \delta \sqrt{n})
$$

where $C=\frac{\sqrt{m}}{\sqrt{m+\left(\sigma_{m} / \sigma\right)^{2}}}=\frac{\sqrt{m}}{\sqrt{m+C_{1}^{2}}}$ and $\Phi($.$) is the standard normal cumulative$ distribution function. The constant $C \in[0,1]$. When $C=1$, the measurement system is free of error. As $C$ decreases, the power of the $\bar{X}$ chart in signaling a process mean shift also decreases. Table 1 brings the values of $C$ for $\mathrm{m}=1$ or 4 and $C_{1}=\sigma_{m} / \sigma=0 ; 0.1 ; 0.3 ; 0.5$ or 1.0.

Table 1. Values of $C$

|  | $C$ |  |
| :---: | :---: | :---: |
| $C_{1}$ | $m=1$ | $m=4$ |
| 0 | 1 | 1 |
| 0.1 | 0.9950 | 0.9988 |
| 0.3 | 0.9578 | 0.9889 |
| 0.5 | 0.8944 | 0.9701 |
| 1.0 | 0.7071 | 0.8944 |

The measurement error reduces when the sample items are measured several times ( $\mathrm{m}>1$ ). For instance, when the variability of the measurement system corresponds to $30 \%$ of the process variability ( $\sigma_{m} / \sigma=0.3$ ) the value of $C$ is 0.9578 or 0.9889 as each sample item is measured once or four times. Figure 1 shows the effect of the measurement error on the $\bar{X}$ chart's performance. For instance, free of measurement error the $\bar{X}$ chart requires, on average, 6.3 samples of size $4(A R L=6.3)$ to signal a shift of one standard deviation $\left(\mu_{1}=\mu_{0} \pm \delta \sigma\right)$. The $A R L$ increases to 17.7 when the variability of the measurement system is the same of the process variability ( $\sigma_{m} / \sigma=1.0$ ). The effect of the measurement error is partially dissipated by measuring each item of the sample several times. According to Figure 2, if each item of the sample is measured four times the $A R L$ reduces from 17.7 to 8.9.


Figure 1. The effect of the measurement error on the $A R L, m=1$ and $n=4$


Figure 2. The effect of the measurement error on the $A R L, m=4$ and $n=4$

## 3. The Effect of the Autocorrelation on the $\bar{X}$ Chart's Performance

In this section, we assume that the observations of the quality characteristic to be monitor fit to a First Order Autoregressive AR (1) model

$$
\begin{equation*}
Y_{t}-\mu=\phi\left(Y_{t-1}-\mu\right)+\varepsilon_{t}, \quad \mathrm{t}=1,2,3, \ldots \tag{2}
\end{equation*}
$$

with the process variance given by

$$
\begin{equation*}
\sigma^{2}=\frac{\sigma_{\varepsilon}^{2}}{1-\phi^{2}} \tag{3}
\end{equation*}
$$

We also assume that the AR (1) model is accurate and the sampling interval is enough long to assure independence among $Y$ values of neighbor samples. The variance of the sample means is

$$
\begin{equation*}
\sigma^{2}(\bar{X})=\frac{\sigma^{2}}{n C_{2}^{2}} \tag{4}
\end{equation*}
$$

The expression of $n C_{2}^{-2}$ is function of $\phi$, see Table 2.
Table 2. The values of $n C_{2}^{-2}$

| $n$ | $n C_{2}^{-2}$ |
| :---: | :---: |
| 2 | $2+2 \phi$ |
| 3 | $3+4 \phi+2 \phi^{2}$ |
| 4 | $4+6 \phi+4 \phi^{2}+2 \phi^{3}$ |
| 5 | $5+8 \phi+6 \phi^{2}+4 \phi^{3}+2 \phi^{4}$ |
| 6 | $6+10 \phi+8 \phi^{2}+6 \phi^{3}+4 \phi^{4}+2 \phi^{5}$ |

The control limits of the $\bar{X}$ chart are

$$
C L=\mu_{0} \pm k \frac{\sigma}{C_{2} \sqrt{n}}
$$

If the process mean shifts to $\mu_{1}=\mu_{0} \pm \delta \sigma$ the $\bar{X}$ chart has the following probability to signal

$$
P=\Phi\left(-k-C_{2} \delta \sqrt{n}\right)+\Phi\left(-k+C_{2} \delta \sqrt{n}\right)
$$

As $C_{2}$ decreases, the power of the $\bar{X}$ chart in signaling a process mean shift also decreases.
Table 3 brings the values of $C_{2}$ for $n=4$ or 5 and $\phi=0 ; 0.2 ; 0.5$ or 0.7 .

Table 3. Values of $C_{2}$

| $\phi$ | $\mathrm{C}_{2}$ |  |
| :---: | :--- | :--- |
|  | $n=4$ | $n=5$ |
| 0 | 1 | 1 |
| 0.2 | 0.86258 | 0.85279 |
| 0.5 | 0.69631 | 0.67040 |
| 0.7 | 0.60729 | 0.56995 |

Figure 3 shows the effect of the autocorrelation on the $\bar{X}$ chart's performance. For instance, with independent observations, the $\bar{X}$ chart requires, on average, 6.3 samples of size 4 (ARL=6.3) to signal a shift of one standard deviation ( $\mu_{1}=\mu_{0} \pm \delta \sigma$ ). The ARL increases to 18.5 when the $Y$ values are described by a First Order Autoregressive AR (1) model with $\phi=0.5$.


Figure 3. The effect of the autocorrelation on the $\bar{X}$ chart performance
The effect of the autocorrelation might be partially dissipated by building up the samples with non-neighbor items, according to the time they were produced. The $A R L$ values reduce as the samples are built by taking one item from the production line and skipping one, two or more before selecting the next, see Figure 4. Taking into account one standard deviation shift on the process mean and skipping two items, the ARL reduces from 18.5 to 8.4.


Figure 4. The effect of skipping items on the $A R L$ values ( $n=4$ and $\phi=0.5$ )

## 4. The Effect of the Measurement Error and Autocorrelation on the $\bar{X}$ Chart's Performance

Measurement error and the autocorrelation between $Y$ values often exist in quality control applications. Both have an adverse effect on the $\bar{X}$ chart's performance. In this section, we assume again an imprecise system of measurement and a quality characteristic $Y$ described by a First Order Autoregressive AR (1) model

$$
\begin{equation*}
Y_{t}-\mu=\phi\left(Y_{t-1}-\mu\right)+\varepsilon_{t}, \quad \mathrm{t}=1,2,3, \ldots \tag{5}
\end{equation*}
$$

The variance of the sample means is now affected by the autocorrelation and by the measurement error

$$
\begin{equation*}
\sigma^{2}(\bar{X})=\frac{\sigma^{2}}{n}\left(\frac{1}{C_{2}^{2}}+\frac{C_{1}^{2}}{m}\right)=\frac{\sigma^{2}}{n C_{3}^{2}} \tag{6}
\end{equation*}
$$

where $C_{1}=\sigma_{m} / \sigma$.
If the process mean shifts to $\mu_{1}=\mu_{0} \pm \delta \sigma$ the $\bar{X}$ chart has the following probability to signal

$$
P=\Phi\left(-k-C_{3} \delta \sqrt{n}\right)+\Phi\left(-k+C_{3} \delta \sqrt{n}\right)
$$

When $C_{3}=1$, the system of measurement is free of error and the $Y$ values are independent. As $C_{3}$ decreases, the power of the $\bar{X}$ chart in signaling a process mean shift also decreases. The values of $C_{3}$ for $\mathrm{m}=1$ or $4, n=4$ or $5, \phi=0.2 ; 0.5$ or 0.7 and $C_{1}=0.3$; 0.5 or 1.0 are in Table 4.

Table 4. Values of $C_{3}$

| $\phi$ | $C_{1}$ |  |  |
| :---: | :---: | :---: | :---: |
|  | 0.3 | 0.5 | 1.0 |
| 0.2 | 0.8351 | 0.7921 | 0.6532 |
| 0.5 | 0.6816 | 0.6576 | 0.5714 |
| 0.7 | 0.5975 | 0.5811 | 0.5191 |

Figure 5 shows the effect of the autocorrelation combined with the measurement error on the $\bar{X}$ chart's performance. For instance, with independent observations and a perfect system of measurement, the $\bar{X}$ chart requires, on average, 6.3 samples of size 4 (ARL=6.3) to signal a shift of one standard deviation ( $\mu_{1}=\mu_{0} \pm \delta \sigma$ ). The ARL increases to 32.5 when the $Y$ values fit to a First Order Autoregressive AR (1) model with $\phi=0.5$ and the variability of the measurement system is the same of the process variability ( $C_{1}=\sigma_{m} / \sigma=1.0$ ).


Figure 5. Effect of the autocorrelation and measurement error on the ARL

$$
(n=4, m=1 \text { e } \phi=0.5)
$$

The joint effect of the autocorrelation and measurement error might be partially dissipated by building up the samples with non-neighbor items, according to the time they were produced and measuring each item of the sample several times. Taking into account one standard deviation shift on the process mean, if $C_{1}=1,0$ and $\phi=0,5$ then building up the samples by taking one item from the production line and skipping two before selecting the next and measuring four times the quality characteristic of each item selected to form the sample, the $A R L$ reduces from 31.6 to only 8.9 , see Figure 6.


Figure 6. Recovering the chart's performance by multiple measurements and skipping items

$$
\left(n=4, C_{1}=1.0 \text { e } \phi=0.5\right)
$$

## 5. Conclusions

In this paper, we studied the joint effect of the autocorrelation and the measurement error on the $\bar{X}$ chart's performance. Additionally, we showed that the undesired effect of the autocorrelation might be reduced by building up the samples with non-neighbor items, according to the time they were produced. Measuring the quality characteristic of each sample item several times reduces the undesired effect of measurement error.

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