An Exact Method to Solve the Optimal Capacitor Placement Problem on Power Distribution Networks

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Abstract

Four decades ago Durán proposed the use of dynamic programming to find the ideal number, best locations and optimal sizes for shunt capacitor banks on radial power distribution feeders. Based on Bellman's Principle of Optimality, Durán's approach was attractive as it provided global optimal solutions for this problem. Conversely, the approach could only address the capacitor allocation problem for feeders without lateral branches, a restriction that shaded the value of his contribution. Herein, the dynamic programming approach is extended with concepts that allow consideration of the main requirements of the capacitor allocation problem, including representation of feeders with lateral branches and different voltage magnitudes along the feeders. As a by-product of the dynamic programming optimization procedure the approach also solves the capacitor control problem, obtaining the best tap adjustment for the switchable capacitive power sources. The key new concepts are the generalization of the definition of *state* proposed by Durán and the projection of the multidimensional information into an equivalent one-dimensional representation. Case studies with real size distribution networks put into perspective the benefits of the Extended Dynamic Programming approach. This approach has the appeal of providing global optimal solutions with a linear time computational complexity, which is similar to the computational complexity of the dynamic programming approach for feeders without lateral branches. Such attributes provide the methodology with a prominent place among the methods to address the shunt capacitor allocation problem on radial distribution networks.

KEYWORDS:power distribution networks, distribution feeders, technical loss reduction, shunt capacitor allocation, optimal capacitor placement and sizing, dynamic programming, extended dynamic programming.

1 Introduction

Distribution networks in a power system connect the distribution substation to customers. They are designed as a set of radial *feeders* rooted at the substations, which are subdivided in primary networks, at the upper level, and secondary networks, at the lower level. Shunt capacitors are placed on the primary networks of distribution feeders to reduce technical losses caused by reactive energy; other potential benefits of capacitors include voltage regulation, released capacity of equipments and deferred expenditure on system expansions. The optimal capacitor allocation problem searches for the best compromise between cost of capacitors and their benefits to a network.

Techniques to unveil the best alternatives for capacitor allocation on distribution feeders have been developed for more than 50 years (Schmidt 1956; Neagle and Samson 1956) (most of these are concerned with capacitor placement for loss reduction and voltage regulation). The early studies proposed approximated models that enabled the application of analytical methods (Schmidt 1956; Neagle and Samson 1956; Cook 1959; Schmill 1965). Durán relied on dynamic programming to address the problem under a more general formulation (Durán 1968), though restricted to feeders without lateral branches (single-ended feeders).

As radial distribution feeders are designed with many lateral branches, with layouts such as the “big trunk”, “feathered”, “multi-branch” and “mixed” (Willis 2004), the merits of Durán’s ideas were hidden by the restriction to single-ended feeders. Furthermore, when the lateral branches are considered, his formulation seemed to require additional state dimensions, in such a way that would preclude the application of his concepts to real scale distribution feeders.

Presently, heuristic methods dominated the scene for optimal capacitor allocation on radial distribution feeders. Baran and Wu, for example, proposed a heuristic method guided by the solution of a mixed integer programming problem (Baran and Wu 1989a; Baran and Wu 1989b); Gallego *et al.* adopted the tabu search heuristic (Gallego et al. 2001); and Mendes *et al.* proposed a hybrid genetic algorithm (Mendes et al. 2005).

This paper champions the revival of dynamic programming. Durán’s restriction to single-ended feeders is overcome and dynamic programming extended to find optimal solutions for the capacitor allocation problem on real-scale multi-branched distribution feeders.

2 Mathematical Formulation

The mathematical formulation is in accordance with the present-day perception of the problem. All the technical and economic benefits of proper capacitor banks allocation can be included in the formulation. However, to avoid unnecessary intricacy in notations, only the compromise between cost of capacitors and loss reduction is considered here.

A graph model is adopted to represent the main entities of a primary distribution network and their interrelationships (Ahuja et al. 1993; Cavellucci and Lyra 1997). When a graph, $\mathcal{G} = [\mathcal{N}, \mathcal{A}]$, represents a primary distribution network for the capacitor optimization problem, *nodes* in set $\mathcal{N}$ represent either bus bars or substations. Substations are the *root nodes* for the distribution feeders that emanates from them. *Arcs* in set $\mathcal{A}$ represent the distribution lines.The paper is concerned with primary distribution networks that operate radially. Figure 1 illustrates the graph model for a typical distribution feeder.

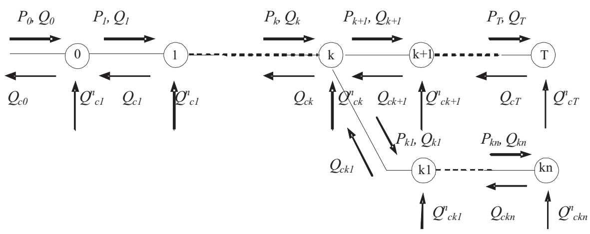

Figure 1: Graph model for a feeder of a primary distribution network

The problem of optimal shunt capacitor allocation on a radial distribution feeder throughout a given planning period, $\mathcal{P}_{CA}$, can be stated as follows:

$$Min_{C_k \in \mathcal{F}}[\sum_{k \in \mathcal{N}} f(C_k) + \alpha_{et} \sum_{t \in T} \tau_t l_t(P,Q,V)] \tag{1}$$

subject to

$$l_t(P,Q,V) = \sum_{k \in N} \sum_{j \in A_k} r_{k+1,j} \frac{(P^t_{k+1,j})^2 + (Q^t_{k+1,j})^2}{(v^t_{k+1,j})^2} \tag{2}$$

$$\forall t \in T \quad \begin{cases} P^t_k = \sum_{j \in A_k} P^t_{k+1,j} + P^t_{L_k} \\ Q^t_k = \sum_{j \in A_k} Q^t_{k+1,j} + Q^t_{L_k} - Q_{C_k} \end{cases} \tag{3}$$

$$\forall t \in T \quad \begin{cases} (v^t_{k+1,j})^2 = (v^t_k)^2 - \\ -2(r_{k+1,j} P^t_{k+1,j} + x_{k+1,j} Q^t_{k+1,j}) \\ \\ \underline{v_k} \le v^t_k \le \overline{v_k} \end{cases} \tag{4}$$

where $\mathcal{F}$ is the set of fixed capacitors for possible installation on the network, $\mathcal{N}$ is the set of nodes in the distribution network, $f(C_k)$ is the time value of the cost of a fixed capacitor C_k that gives the reactive power Q_{C_k}, α_{et} is the value of energy during interval t, τ_t is the duration of interval t, T is the set of time intervals, $l_t(P,Q,V)$ is the power loss in the network during interval t, A_k is the set of arcs with origin at node k (emanating from node k), $r_{k+1,j}$ is the resistance of the line represented by arc $k+1, j$, $P^t_{k+1,j}$ is the total active power flow in arc $k+1, j$ during the time interval t, $Q^t_{k+1,j}$ is the total reactive power flow in arc $k+1, j$ during the time interval t, $P^t_{L_k}$ and $Q^t_{L_k}$ are, respectively, active and reactive power loads at the node k during the time t. The variables v^t_k (or $v^t_{k+1,j}$) represent the voltage values in node k (or $k+1, j$) during the time interval t, $\underline{v_k}$ and $\overline{v_k}$ are, respectively, lower and upper bounds for v^t_k.

Since distinct active and reactive power loads are considered for each node k and each time period t, all possible load levels combinations can be considered with this formulation of the problem.

The double indexes $k+1, j$ are adopted here to stress the point that the nodes $k+1, j$ are *successors* of the node k in the *tree* that represents the feeder (or conversely, that k is a *predecessor* of all nodes $k+1, j$).

The power flow and voltage magnitude equations in (3) and (4) are the simplified DistFlow branch equations presented in (Baran and Wu 1989c), but generalized to consider multi-ended feeders. There are no restrictions in adopting the more detailed DistFlow branch equations proposed in (Baran and Wu 1989c), but the simplified equations allow a leaner presentation which helps to focus on the main points of the paper.

Durán (1968) proposed a dynamic programming approach to address a simplified version of the problem $\mathcal{P}_{CA}$, for which the distribution feeders are single-ended (in other words, do not have lateral branches) and voltages are supposed close to their specified values—in other words, $v_k = 1$ *p.u.*

(Gonen 2008). The essence of this approach is briefly described in the next section, with the necessary adaptations to address the formulation for the problem $\mathcal{P}_{CA}$ presented above, which is more in accordance with the present-day perception of the problem. These ideas are then generalized to consider multi-ended feeders (feeders with lateral branches) and instances for which voltage values can not be adequately approximated by $v_k = 1$ *p.u.*.

3 Durán's Dynamic Programming Approach

Using the language of dynamic programming (DP), Durán (1968) associated the nodes of a single-ended distribution network to the *stages*, the capacitive reactive power injected at a node k to the *control variable* at the node ($u_k = Q_{C_k}$) and the total capacitive power flowing in the arc immediately upstream (toward the source) from a node k to the *state* at this node (x_k). Since feeders are assumed to be single-ended, *states* and *control variables* satisfy the following *dynamic equation*:

$$x_k = x_{k+1} + u_k \tag{5}$$

The *elementary cost* (the cost over the next stage) for the control action u_k applied at a state x_k is defined as $e_k(x_k, u_k)$ (remember that v_k is assumed to be 1 *p.u.*),

$$e_k(x_k, u_k) = f(u_k) + \alpha_{et} \sum_{t \in T} \tau_t r_k [(P_k^t)^2 + (Q_k^t - u_k)^2] \tag{6}$$

Therefore, the iterative functional equation for the solution of the problem $\mathcal{P}_{CA}$ by DP can be stated as follows:

$$\begin{gathered} F(x_k) = Min_{u_k}[e_k(x_k, u_k) + F(x_{k+1})] \\ x_{k+1} = x_k - u_k \end{gathered} \tag{7}$$

where $F(x_k)$ is the *optimal cost function* from the state x_k at the stage k, which represents the best allocation of a total capacitive power x_k, from k until the end of the feeder. The minimization procedure also gives the optimal control variable for the state x_k, $u_k^*(x_k)$; in other words, it provides the best amount of capacitive power to be placed at the node k, if a total amount x_k of reactive power is installed from node k down to the end of the feeder. Of course, the recursive optimization process must meet Eq. (3).

The iterative functional equation is initialized at the end node of the feeder (say, at its leaf),

$$F(x_n) = e_n(x_n, u_n), \quad u_n = x_n \tag{8}$$

When the recursive backward computation process with evaluation of Eq. (7) reaches the substation (stage 1, the root of the feeder), a solution to the problem $\mathcal{P}_{CA}$, $F(x_1^*)$, can be easily obtained by searching over all values of $F(x_1)$,

$$F(x_1^*) = Min_{x_1}[F(x_1)] \tag{9}$$

The optimal placement and size of capacitors in the feeder (the "optimal trajectory") is recovered with a recursive forward computation procedure, from the root to the leaf of the feeder,

$$x_2^* = x_1^* - u_1^*(x_1^*), \quad x_{k+1}^* = x_k^* - u_k^*(x_k^*) \tag{10}$$

Figure 2 illustrates this classical DP approach tailored to the solution of the problem, $\mathcal{P}_{CA}$.

If there is only one arc downstream from each node, this DP approach requires only one-dimensional *states*. If there is more than one arc emanating from some of the nodes, additional

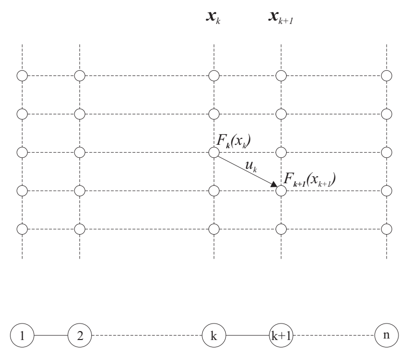

Figure 2: DP approach to single-ended feeder

state dimensions appear to be needed, one for each lateral branch. With such a representation, using DP to compute a solution for real feeders with many lateral branches would meet "the curse of dimensionality", a term coined to express the fact that the computational effort grows exponentially with the dimension of the state vector—a property that turns the problem intractable (Garey and Johnson 1979). This might be the reason for which DP has been forgotten as a technique to solve the capacitor allocation problem.

4 The Extended Dynamic Programming Approach

In essence, instead of increasing the dimension of *state* variables for each additional downstream branch, the Extended Dynamic Programming (EDP) approach proposes simple auxiliary optimization problems that projects the problem, $\mathcal{P}_{CA}$, into a one-dimensional DP representation.

4.1 Key Concepts

To simplify presentation, it will still be assumed that voltage magnitudes can be adequately approximated by $v_k = 1$ *p.u.*. This assumption will be dropped later, in Section 4.2, which discusses instances for which voltage monitoring is necessary.

The main concepts proposed by Durán (1968) to embed the problem $\mathcal{P}_{CA}$ into a dynamic programming solution framework are still valuable when distribution feeders are not single-ended. Namely, nodes in distribution feeders are associated with *stages* and the capacitive reactive power (Q_{C_k}) injected at a node k is associated with the *control variable* at the node ($u_k = Q_{C_k}$). In addition, the *elementary cost* is the same—the influence of lateral branches will be considered in Eq. (11).

For a while, suppose that the *state* (x_k) is still the total capacitive power flowing in the arc immediately upstream from the node k. Note that Eq. (5) is no more true if more than an arc emanates from node k, as illustrated in Fig. 3. The inclusion of an additional dimension in the state is an alternative to deal with this problem; the state at each stage $k+1$ would be a vector with as many dimensions as the number of branches emanating from the node. If, for instance, two branches emanate from a node k, the dynamic equation for the node would be $x_k = x_{k+1,1} + x_{k+1,2} + u_k$, instead of Eq. (5). However, in the forward process to recover the optimal trajectory, it would only be possible to recover the sum $x_{k+1,1} + x_{k+1,2}$, leaving an indetermination regarding the value of each $x_{k+1,j}$.

A first thought to overcome the indetermination about the value of each $x_{k+1,j}$ under the framework of conventional dynamic programming would be to keep the additional state dimension in the backward process, up to the root. As such, a state dimension would be necessary to store information about each lateral branch of the feeder. However, it is not advantageous to delve further in

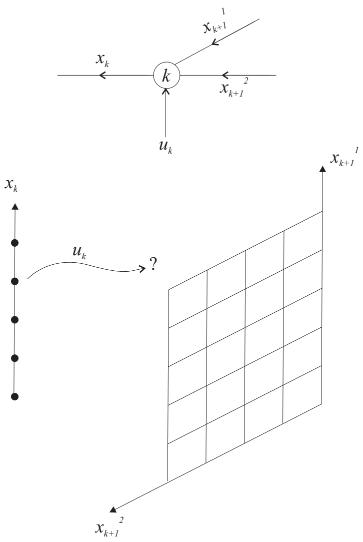

Figure 3: Does the Problem Need a Multidimensional DP Algorithm?

this direction. The multidimensional state would be necessary only to uncover the optimal share of capacitive power for each branch emanating from a node k. The "optimal share" is the clue.

The key concept to allow the consideration of lateral branches at a node in a DP solution framework is to redefine the *state* at a node k as "the total capacitive power flowing in the arc immediately upstream, optimally divided among all branches emanating from k". The essence is indeed as simple as this, but it is necessary to fathom the consequences of this new concept.

1. How to find the optimal share of capacitive power for each branch emanating from a node?
2. How are the DP backward and forward processes affected by the new concept?

Auxiliary optimization problems that project the multidimensional informations into one-dimension and the definition of a vector-valued *optimal share function*, for each node with lateral branches, will open the path to cope with both the questions above. Before probing further, note that the new definition of *state* reduces to the former if only one state emanates from each node (when the feeder is single-ended).

Formally, the following auxiliary optimization problems must be solved during the dynamic programming backward optimization process, for each node k with lateral branches:

$$\begin{gathered} F(x_{k+1}) = Min_{\forall x_{k+1,j} \in J}[\textstyle\sum_{j\in J} F(x_{k+1,j})] \\ x_{k+1} = \textstyle\sum_{j\in J} x_{k+1,j} \end{gathered} \tag{11}$$

where J is the set of indexes of lateral branches at the node k and $x_{k+1,j}$ is the state at node "$k+1$", branch j ($j \in J$). These problems are easily solved with the usual enumeration process of dynamic programming.

The solution process also gives the vector-valued *optimal share function* O_{k+1},

$$O_{k+1} : \Re \longrightarrow \Re^{\jmath}$$

where $\jmath$ is the number of branches emanating from node k (the cardinality of set J).

$$O_{k+1}(x_{k+1}) = \begin{pmatrix} x^o_{k+1,1} \\ x^o_{k+1,2} \\ \vdots \\ x^o_{k+1,\jmath} \end{pmatrix} \tag{12}$$

The points $x^o_{k+1,1}, x^o_{k+1,2}, \ldots, x^o_{k+1,j}$ of $O_{k+1}(x_{k+1})$ are the values for which Eq. (11) achieves the minimum value, on each hyperplane $x_{k+1} = \sum_{j \in J} x_{k+1,j}$ (Bertsekas 1995).

In solving Problems (11), the multidimensional informations stored in the set of *states*, associated with the branches emanating from the node k are projected into an equivalent one-dimensional optimal representation. This procedure is equivalent to creating a dummy $k+1$ node and correspondent x_{k+1} *state* that embody the multidimensional informations. The optimal partition of capacitive power among the arcs that emanate from a node and the associated *optimal cost function* $F(x_{k+1})$ are found and stored at the points of the feeder where they are necessary for the DP backward optimization process and forward recovering of the optimal trajectory. No additional dimension for the *state* is necessary to achieve this synthesis and nothing is lost, compared with a conventional multidimensional dynamic programming approach. Figure 4 illustrates the projection procedure for an example with two lateral branches emanating from a node k.

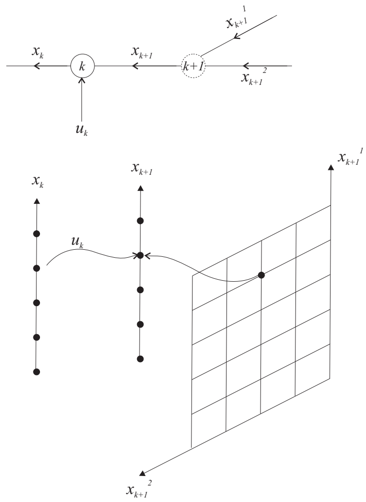

Figure 4: Dealing with lateral branches

The recursive dynamic programming process starts at the leaves of the feeder. For every node k with lateral branches, a projection problem is solved and an *optimal share function* (O_{k+1}) is determined. Analogous to single-ended feeders, the Iterative Functional Equation (7) is solved at each node k.

When recovering the optimal trajectory in the forward procedure, for nodes without lateral branches, x_{k+1} is obtained in the same way as single-ended feeders, applying Eq. (10). When more than an arc emanates from a node j, each optimal share, $x^o_{k+1,j}$, is obtained by applying the *optimal share function* at x_{k+1}, $O_{k+1}(x_{k+1})$. The recovery process continues, starting from each $x^o_{k+1,j}$, until it reaches all leaves of the feeder.

4.2 Considering Different Voltage Magnitudes ($v_k \neq 1$ *p.u.*)

The main points to be addressed in order to take into account differences in voltage magnitudes in multi-branched feeders are how to compute efficiently the voltages values in a multi-branched feeder and, most importantly, if the solution of Problem $\mathcal{P}_{CA}$ changes when different voltage values are taken into account. The graph representation of the primary distribution feeder provides help, once again; both points are easily handled.

Consider that the optimal capacitor allocation problem, $\mathcal{P}_{CA}$, has been solved with the EDP approach. Consequently, all power flows for the feeder are known. Now, in order to compute efficiently the voltages values in a multi-branched feeder, it sufices to visit all nodes of the feeder with a *preorder traversal* of the graph (Ahuja et al. 1993), evaluating the voltage values at each node by Eq. (4); in other words, it suffices to compute voltage magnitudes following the *thread(j)* index (discussed in the previous section).

Voltage regulators, if installed on the network, have to be taken into consideration when computing voltage magnitudes along the feeder. When a voltage regulator is installed in a node k of the feeder, the regulated value of the voltage ($\mathcal{V}_k^t$) should be obtained after the computation of v_k^t,

$$\mathcal{V}_k^t = g_k(v_k^t)$$

where the function g_k represents the operation rule of the voltage regulator. After the computation of $\mathcal{V}_k^t$, the computation of voltages goes on for the nodes downstream from node k, using Eq. (4) and considering $\mathcal{V}_k^t$ as the voltage at node k.

Having knowledge of the voltage magnitudes after the allocation of capacitors, the problem $\mathcal{P}_{CA}$ is solved again, but now substituting the *elementary cost* $e_k(x_k, u_k, v_k)$ given by Eq. (13), which considers the influence of different voltage magnitudes, for the *elementary cost* given by Eq. (6).

$$\begin{aligned} e_k(x_k, u_k, v_k) &= f(u_k) + \\ &+ \alpha_{et} \sum_{t \in T} \tau_t r_k \frac{[(P_k^t)^2 + (Q_k^t - u_k)^2]}{(v_k^t)^2} \end{aligned} \tag{13}$$

If the placement and size of capacitors remain the same, the solution found is optimal. Otherwise, if the placement or the size of at least one capacitor changes, voltage magnitudes should be recalculated and the problem solved again. The process is repeated until the solution remains unchanged.

Thus, the method to consider different voltage magnitudes along the feeder goes as follows.

Step 1. Solve problem $\mathcal{P}_{CA}$ with EDP using Eq. (6) to compute the *elementary cost* (considering $v_k = 1$ *p.u.*);

Step 2. Compute the voltage value for each node of the feeder, by Eq. (4);

Step 3. Solve problem $\mathcal{P}_{CA}$ with EDP using Eq. (13) to compute the *elementary cost* (considering $v_k \neq 1$ *p.u.*);

Step 4. Compare the current capacitor allocation solution with the previous one. If the solution is different, return to *Step 2*. Otherwise, stop; the current solution is optimal.

Note that from a rigorous mathematical stand point this iterative procedure can only assure local optimality for the actual state of voltage values. The physics of the problem provides the additional argument for global optimality; as voltage magnitudes change smoothly along the feeders, there are no extemporaneous voltage magnitude profiles that would allow solutions to be too far away from each other.

5 The Complexity of Extended Dynamic Programming

Call Q_{max} the number of possible *controls* for a given node, which correspond to the number of possible capacitors that could be installed at the node. Also, call N the number of nodes of the feeder and J_{max} the maximum number of lateral branches at a node—in real instances, usually J_{max} does not exceed 3 (Willis 2004).

The main steps for solving any problem $\mathcal{P}_{CA}$ with EDP are the following:

1. solve Eq. (8) for each *leaf* node: the number of computations to solve Eq. (8) for all *leaf* nodes is bounded by $c_1 N_L Q_{max}$, where c_1 is a constant and N_L is the number of *leaf* nodes. Since $N_L < N$, then $c_1 N Q_{max}$ remains a valid upper bound;
2. solve the DP Iterative Functional Equation (7) for each non *leaf* node: the total number of computations to solve the Iterative Functional Equation (7) for all nodes is bounded by $c_2 N {Q_{max}}^2$, where c_2 is a constant;

3. solve the Projection Equation (11) for each node with lateral branches: the total number of evaluations to compute the Projection Equation (11) is bounded by $c_3 N_B {Q_{max}}^{J_{max}}$, where c_3 is a constant and N_B is the maximum number of lateral branches. Since $N_B < N$, the total number of evaluations is, for better reasons, bounded by $c_3 N {Q_{max}}^{J_{max}}$;
4. recover the "optimal trajectory", using Eq. (10) and the *optimal share functions* given by Eq. (12): the forward process to recover the "optimal trajectory" computes one *state* and one *control* for each node, therefore is bounded by $c_0 N$ computations, where c_0 is a constant.

Thus, the number of computations to solve the capacitor allocation problem $\mathcal{P}_{CA}$ with EDP is bounded by

$$N \cdot [c_0 + c_1 Q_{max} + c_2 {Q_{max}}^2 + c_3 {Q_{max}}^{J_{max}}] \in O(N {Q_{max}}^{J_{max}}).$$

Real instances have numbers of controls (Q_{max}) and maximum number of lateral branches (J_{max}) limited by small integers. Therefore, it is possible to consider the quantity inside the brackets as a constant c. Under this assumption, the EDP algorithm has linear complexity, described by $cN \in O(N)$.

6 Considering Switched Capacitors

It is simple to study the optimal allocation of switched capacitors instead of fixed capacitors with the EDP approach discussed so far. Indeed, the only necessary change is to consider the possibility of the best tap adjustment for u_k, when computing the *elementary cost* in Eq. (6) and (13).

A more interesting study is to consider the optimal allocation of both fixed and switched capacitors. Here is a case where two dimensions for the *state* at a node k is necessary: one to represent the total fixed capacitive power flowing in the arc immediately upstream, optimally divided among all branches emanating from k; the other to represent the total switchable capacitive power flowing in the arc immediately upstream, optimally divided among all branches emanating from k.

All concepts presented here can be easily extended to address the generalized problem of placing both fixed and switched capacitors on the network. Also, it is simple to verify that the EDP algorithm for the problems still have linear complexity described by $cN \in O(N)$, with a larger value for the constant c. Programming, however, is harder (much harder).

7 Case Studies

Four networks from different distribution systems were used to evaluate the EDP approach. The first one (A) is the 70-bus test system adopted by Baran and Wu (Baran and Wu 1989a). The other networks are real large scale instances of distribution systems in the state of São Paulo (Brazil): a 2645-bus system (B), a 6246-bus system (C) and a 7500-bus system (D). The feeders in system B, C and D have significant voltage drops and power losses.

Table 1: Main Data for the Distribution Networks

Network	**Number of Nodes**	**Number of Feeders**	**Nodes w/l Branches**	**Total MW**	**Total MVAR**
A	70	1	7	3.80	2.69
B	2 645	11	511	42.76	20.72
C	6 246	30	1 143	131.97	66.90
D	7 500	10	1 304	61.60	29.57

The capacities and costs of the capacitor banks are summarized in Table 2. Other parameters used were an energy value of 60 US$/MWh and a 5 year payback period for investments in capacitors, with interest of 15%. Two different load scenarios are considered: the scenario adopted for allocation of only fixed capacitor banks consider a single time interval (in other words, $T = 1$ in the problem $\mathcal{P}_{CA}$); the scenario adopted for studies of allocation of both fixed and switched capacitor banks uses load profiles with five different levels (0.3 1.0 0.8 0.8 0.6), with duration of (6 6 6 3 3) hours, respectively.

Table 2: Capacitor Banks

Capacity (kVAr)	**Fixed Bank Cost (US \$)**	**Switched Bank Cost (US \$)**
150	3 494	4 494
300	3 553	4 553
450	3 628	4 628
600	4 026	5 026
900	4 992	5 992
1 200	5 958	6 958

Two different versions of the EDP algorithm were considered: **EDP-1** which considers $v_k = 1\ p.u.$ and **EDP-2** which considers $v_k \neq 1\ p.u.$ (using the method described in Section 4.2);

All algorithms were coded in C++ using the *GCC 4.3* compiler without optimization flags. Computational tests were run on a PC with a Core 2 Quad 3.0 GHz processor and 4 GB of RAM, running under GNU/Linux.

Table 3 presents the energy savings obtained, with the EDP-1 and EDP-2 versions of the EDP approach. The computation of initial and final losses in both tables considers voltage drops along the feeders. The “Time” columns give the execution times, in seconds.

Table 3: Results for the two approaches

Network	**Initial Losses (kW)**	**EDP-1 Losses kW**	**EDP-1 Savings %**	**EDP-1 Time Sec**	**EDP-2 Losses kW**	**EDP-2 Savings %**	**EDP-2 Time Sec**
A	39.42	28.65	27.4	0.008	28.65	27.4	0.008
B	1 261.84	991.62	21.5	0.256	979.92	21.5	0.364
C	3 428.99	2 738.06	20.1	0.992	2 706.19	21.1	1.396
D	3 168.98	2 345.86	26.0	1.280	2 281.79	28.0	1.720

Table 4 presents the economic benefits of the solutions using the EDP-1 and EDP-2 algorithm, respectively, considering the value of energy savings and the cost of capacitors. Both tables also give the total capacitive power installed on the networks.

Table 4: Economic Benefits of the EDP-1 and EDP-2 approaches

Network	**Initial Cost (US\$)**	**EDP-1 Cost (US\$)**	**EDP-1 Savings (%)**	**EDP-1 Total (kVAr)**	**EDP-2 Cost (US\$)**	**EDP-2 Savings (%)**	**EDP-2 Total (kVAr)**
A	20 721	16 548	21.2	900	16 548	21.2	900
B	663 224	546 563	17.6	15 300	542 306	18.3	16 650
C	1 802 280	1 512 260	16.1	44 250	1 502 670	16.7	49 050
D	1 665 610	1 271 030	23.7	22 050	1 245 870	25.3	28 500

The results presented in Table 3 and Table 4, allow the following observations:

1. EDP is a feasible approach to find the best places and optimal sizes of shunt capacitor banks on radial real scale distribution feeders;
2. When voltage drops are significant, their representation can improve the decisions about capacitor allocation;
3. Processing times for both EDP algorithms are very short, even for large feeders;

4. Processing times are approximately proportional to the number of nodes in the feeders, confirming the linear complexity;
5. The processing times for applying the EDP-1 and EDP-2 approaches to network A are roughly the same, because for this small instance the set up time of the program is significant, compared to the total time required to solve the problem;
6. The processing times for applying the EDP-2 approach to the networks B,C, and D are a little shorter than twice the processing times for applying the EDP-1 approach; this occurs because EDP-2 converged with two iterations (and because the total processing time includes the set up time).

8 Conclusions

Four decades ago Durán proposed the use of dynamic programming to address the capacitor allocation problem (Durán 1968). However, his approach could only address the problem for single-ended feeders, a limitation that hid the value of his ideas.

The Extended Dynamic Programming (EDP) approach developed in this paper allows to lift the simplifying assumption of single-ended feeders and to consider other important aspects of the capacitor allocation problem on radial distribution networks:

1. Finding a global optimal solution for the problem;
2. Take into account differences in voltage magnitudes and voltage regulation;
3. Contemplate all possible load level combinations (for each time period and each node of the network);
4. Considering switched capacitors.

Case studies with real size distribution networks certified EDP with field tests. Bellman's Principle of Optimality (Bellman 1957) provides the formal background to assure that the approach unveils global optimal solutions.

A key concept for the design of EDP was the new definition of *state* at a node, as the total capacitive power optimally divided among all branches emanating from the node. Auxiliary optimization problems that projected multidimensional informations into one-dimension equivalents and a vector-valued *optimal share function* for each node with lateral branches were the complementary conceptual tools employed.

The EDP algorithm has a linear time complexity described by cN (see Section 5), where c is a constant and N is the number of nodes in the network. The framework of computational complexity (Papadimitriou and Steiglitz 1982) allows two additional conclusions to be drawn from this property.

- An internal conclusion, to do with the algorithm: under the assumptions of Problem $\mathcal{P}_{CA}$, EDP is an "efficient algorithm" (in a formal sense) to address the capacitor allocation problem.

- An external conclusion, to do with the knowledge EDP brings about the nature of the problem: since there is an "efficient algorithm" to solve the capacitor allocation problem on radial feeders, it can not be considered a "difficult problem" (for which heuristic methods would be required).

To sum up, the Extended Dynamic Programming approach has a set of qualities that allows it a prominent role in addressing the shunt capacitor allocation problem on radial distribution networks.

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