AN INEXACT PROJECTION METHOD FOR SOLVING VARIATIONAL INEQUALITIES

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ABSTRACT
We analyze an inexact projection method for solving the variational inequality problem. The convergence of the generated sequence is proved under mild assumptions. A numerical experiment is reported.

KEYWORDS. variational inequalities, pseudomonotone$_*$ operator, projection methods, Mathematical programming.

1 Introduction

Let C be a nonempty closed convex subset of $\mathbb{R}^n$ and let $T\colon \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ be a point-to-set operator. We consider the *Variational Inequality Problem* defined by:

$$VIP(T,C)\begin{cases} \text{Find } \ x^* \in \ C \ \text{ such that there exists } \ u^* \in T(x^*) \ \text{ with} \\ \langle u^*, x - x^*\rangle \geq 0 \quad \forall \ x \ \in C. \end{cases} \tag{1}$$

The variational inequality problem was introduced by Hartman and Stampacchia (1966). This formulation has been used to study several problems in transportation planning, regional science, socio-economic analysis, energy modeling, and game theory, see, for instance, Facchinei and Pang (2003) and references therein.

In this paper, we study a relaxed version of the algorithm given in Anh et al (2008). We consider inexact projections onto the constraint set C and the step size is defined by a divergent series. We prove the convergence of the sequence generated by the algorithm under mild assumptions. We include a numerical experiment to illustrate the behavior of our method.

This work is organized as follows. In Section 2, we give some basic facts. In Section 3, we define the algorithm and we analyze its convergence. Finally, we report a preliminary numerical experience.

2 Preliminaries

In this work we assume that the operator T is bounded on bounded subsets of C and it has a sequentially closed graph, that is, if $\{(x^k, u^k) \ : \ u^k \in T(x^k)\}$ converges to (x, u), then, $u \in T(x)$.

We denote by $\mathrm{S}(T, C)$ the solution set of $\mathrm{VIP}(T, C)$.

In the following, we recall some necessary definitions and we give some basic results.

Definition 1 *A point-to-set operator $T\colon \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ is called:*

- *pseudomonotone on C if for every $x, y \in C$ and every $u \in T(x)$, $v \in T(y)$, the following implications holds:*

$$\langle u, y - x\rangle \geq 0 \ \ \Rightarrow \ \ \langle v, y - x\rangle \geq 0. \tag{2}$$

- *pseudomonotone$_*$ on C, if it is pseudomonotone and for every $x, y \in C$ and $u \in T(x), v \in T(y)$,*

$$\langle u, y - x\rangle = \langle v, y - x\rangle = 0 \ \ \Rightarrow \ \ v \in T(x), u \in T(y). \tag{3}$$

Remark 1 *The class of pseudomonotone$_*$ operators is significantly larger than the class of paramonotone maps. As an example, the Clarke subdifferential of a locally Lipschitz pseudoconvex function is pseudomonotone$_*$, see Hadjisavvas and Schaible (2009) and references therein.*

Example 1 *Let $n = 1$, $T(x) = x^2$. T is pseudomonotone$_*$ on $C = \mathbb{R}$ and bounded on bounded subsets of C. Furthermore, T is a nonmonotone operator and it has a sequentially closed graph.*

Proposition 1 *Assume that T is a pseudomonotone$_*$ operator on C. Then, for every $x^* \in S(T,C)$, $\overline{x} \in C$ and $\overline{u} \in T(\overline{x})$ it holds*

$$\langle \overline{u}, x^* - \overline{x} \rangle = 0 \;\;\Rightarrow\;\; \overline{x} \in S(T,C). \tag{4}$$

Proof:
Let $x^* \in S(T,C)$, $\overline{x} \in C$ and $\overline{u} \in T(\overline{x})$ such that

$$\langle \overline{u}, x^* - \overline{x} \rangle = 0. \tag{5}$$

Since $x^* \in S(T,C)$ there exists $u^* \in T(x^*)$ such that

$$\langle u^*, \overline{x} - x^* \rangle \geq 0. \tag{6}$$

On the other hand by (5) and (2) we have

$$\langle u^*, \overline{x} - x^* \rangle \leq 0. \tag{7}$$

Therefore, from (6) and (7) it results

$$\langle u^*, \overline{x} - x^* \rangle = 0. \tag{8}$$

Hence, by (3) we have that $u^* \in T(\overline{x})$.
Consequently, for every $y \in C$ it follows

$$\begin{aligned} \langle u^*, y - \overline{x} \rangle &= \langle u^*, y - x^* \rangle + \langle u^*, x^* - \overline{x} \rangle \\ &= \langle u^*, y - x^* \rangle \\ &\geq 0. \end{aligned}$$

that is, $\overline{x} \in S(T,C)$. The proof is complete. □

The following well-known property will be a useful tool.

Lemma 1 *Let $\{a_k\}$ and $\{b_k\}$ be nonnegative sequences of real numbers satisfying $a_{k+1} \leq a_k + b_k$ and such that $\sum_{k=1}^{+\infty} b_k < +\infty$. Then the sequence $\{a_k\}$ converges.*

3 A projection method for variational inequalities

Let $\underline{\rho} > 0$ and let $\{\rho_k\} \subset (\underline{\rho}, +\infty)$, $\{\beta_k\} \subset (0, +\infty)$, $\{\xi_k\}$, $\{\epsilon_k\}$ be sequences of nonnegative parameters such that

$$\sum \frac{\beta_k}{\rho_k} = +\infty, \quad \sum \beta_k^2 < +\infty, \; \sum \xi_k < +\infty, \sum \frac{\beta_k \epsilon_k}{\rho_k} < +\infty. \tag{9}$$

3.1 Algorithm PMVI

step 0: Choose $x^0 \in C$. Set $k = 0$.
step 1: Let $x^k \in C$. Obtain $u^k \in T(x^k)$.
step 2: Find $w^k \in \mathbb{R}^n$ such that:

$$\langle w^k + u^k, y - x^k \rangle \geq -\epsilon_k \;\; \forall \;\; y \in C, \tag{10}$$

and define

$$\alpha_k = \frac{\beta_k}{\gamma_k} \quad \text{where} \quad \gamma_k = \max\{\rho_k, \|w^k\|\}. \tag{11}$$

step 3: Compute $x^{k+1} \in C$ such that:

$$\langle x^{k+1} - x^k - \alpha_k w^k, x - x^{k+1}\rangle \geq -\xi_k \ \ \forall \ \ x \in C. \tag{12}$$

Let us note that if $\xi_k = \epsilon_k = 0$ and $\gamma_k = 1$ for all $k \in \mathbb{N}$ the algorithm (PMVI) becomes the algorithm 2.1 given in Anh et al (2007). Clearly, (10) is verified, if we consider $w^k = -u^k$.

3.2 Convergence analysis

Throughout this work we assume the following assumptions:

A1. The solution set $S(T, C)$ of Problem $VIP(T, C)$ is nonempty;

A2. T is pseudomonotone$_*$.

We observe that condition A2 is weaker than the condition of strongly monotonicity of the operator F used in (Anh et al, 2007).

Now, we are in position to establish our convergence results for VIP(T, C).

Lemma 2 *We have*

$$\|x^{k+1} - x^k\| \leq \frac{\beta_k + \sqrt{\beta_k^2 + 4\xi_k}}{2} \qquad \forall k \in \mathbb{N}. \tag{13}$$

Proof:
By taking $x = x^k$ in (12) we get

$$\langle x^{k+1} - x^k, x^{k+1} - x^k\rangle \leq \langle \alpha_k w^k, x^{k+1} - x^k\rangle + \xi_k. \tag{14}$$

Combining (14) and the Cauchy-Schwarz inequality, we have

$$\begin{array}{rcl} \|x^{k+1} - x^k\|^2 & = & \langle x^{k+1} - x^k, x^{k+1} - x^k\rangle \\ & \leq & \langle \alpha_k w^k, x^{k+1} - x^k\rangle + \xi_k \\ & \leq & \beta_k\|x^{k+1} - x^k\| + \xi_k \end{array} \tag{15}$$

Hence, we complete the proof by considering the quadratic function $s(\theta) = \theta^2 - \beta\theta - \xi$, with $\theta = \|x^{k+1} - x^k\|$. □

Proposition 2 *Assume that A1 is verified. Let $x^* \in S(T, C)$. Then, for all $k \in \mathbb{N}$, the following assertion holds*

$$\|x^{k+1} - x^*\|^2 \leq \|x^k - x^*\|^2 + 2\alpha_k\langle u^k, x^* - x^k\rangle + \delta_k, \tag{16}$$

where $\delta_k = \beta_k\sqrt{\beta_k^2 + 4\xi_k} + \beta_k^2 + 2(\alpha_k\epsilon_k + \xi_k)$.

Proof:
We have

$$
\begin{array}{rcl}
\|x^{k+1}-x^*\|^2 & = & \|x^{k+1}-x^k+x^k-x^*\|^2 \\
& = & \|x^{k+1}-x^k\|^2+\|x^k-x^*\|^2+2\langle x^{k+1}-x^k, x^k-x^*\rangle \\
& = & \|x^k-x^*\|^2-\|x^{k+1}-x^k\|^2+2\langle x^k-x^{k+1}, x^*-x^{k+1}\rangle \\
& \leq & \|x^k-x^*\|^2+2\langle x^k-x^{k+1}, x^*-x^{k+1}\rangle \\
& \leq & \|x^k-x^*\|^2+2\left(\alpha_k\langle w^k, x^{k+1}-x^*\rangle+\xi_k\right) \\
& = & \|x^k-x^*\|^2+2\alpha_k\left(\langle w^k, x^k-x^*\rangle+\langle w^k, x^{k+1}-x^k\rangle\right)+2\xi_k \\
& \leq & \|x^k-x^*\|^2+2\alpha_k\langle u^k, x^*-x^k\rangle \\
& + & 2\left(\beta_k\|x^{k+1}-x^k\|+\alpha_k\epsilon_k+\xi_k\right) \\
& \leq & \|x^k-x^*\|^2+2\alpha_k\langle u^k, x^*-x^k\rangle \\
& + & \beta_k\sqrt{\beta_k^2+4\xi_k}+\beta_k^2+2\left(\alpha_k\epsilon_k+\xi_k\right),
\end{array}
\tag{17}
$$

where second inequality comes from (12), the third one is obtained from (10), (11) and the Cauchy Schwartz inequality, and we get the last inequality from Lemma 2. □

Theorem 1 *Assume that A1 and A2 are verified. Then,*
(i.) $\{\|x^k-x^*\|^2\}$ is convergent, for all $x^* \in S(T, C)$;
(ii.) $\{x^k\}$ is bounded.

Proof:
$(i.)$ Let $k \in \mathbb{N}$, from A2 we get that $\langle u^k, x^*-x^k\rangle \leq 0$. Therefore, by Proposition 2 we have that

$$\|x^{k+1}-x^*\|^2 \leq \|x^k-x^*\|^2+\delta_k, \tag{18}$$

where $\delta_k = \beta_k\sqrt{\beta_k^2+4\xi_k}+\beta_k^2+2\left(\alpha_k\epsilon_k+\xi_k\right)$.
Since, $ab \leq \frac{1}{2}(a^2+b^2)$, for all $a, b \in \mathbb{R}$ we get

$$\beta_k\sqrt{\beta_k^2+4\xi_k} \leq \frac{1}{2}(2\beta_k^2+4\xi_k).$$

Therefore, by (9) we obtain that

$$\sum_{k=0}^{\infty}\delta_k < +\infty.$$

Hence, the convergence of the sequence $\{\|x^k-x^*\|^2\}$ follows from Lemma 1.
$(ii.)$ It is a direct consequence of (i). □

Theorem 2 *Suppose that A1 and A2 are verified. In addition, assume that the sequence $\{w^k\}$ is bounded. Then, the whole sequence $\{x^k\}$ converges to a solution of VIP(T, C).*

Proof:
Let $\{x^k\}$ and $\{u^k\}$ be sequences generated by algorithm (PMVI) and let $x^* \in S(T, C)$.

Firstly, we prove that $\limsup_{k\to+\infty}\langle u^k, x^* - x^k\rangle = 0$.
By applying A2 to $x = x^k$ and by Theorem 1 we have

$$0 \le 2\alpha_k\langle u^k, x^k - x^*\rangle \le \|x^k - x^*\|^2 - \|x^{k+1} - x^*\|^2 + \delta_k, \tag{19}$$

where $\sum_{k=k_0}^{+\infty}\delta_k < +\infty$.
Hence,

$$\begin{aligned} 0 &\le 2\textstyle\sum_{k=k_0}^{m}\alpha_k\langle u^k, x^k - x^*\rangle \\ &\le \|x^k - x^*\|^2 - \|x^{k+1} - x^*\|^2 + \textstyle\sum_{k=k_0}^{m}\delta_k, \end{aligned} \tag{20}$$

by taking limits with $m \to +\infty$ we obtain that

$$0 \le \sum_{k=k_0}^{+\infty}\alpha_k\langle u^k, x^k - x^*\rangle \le \|x^{k_0} - x^*\|^2 + \sum_{k=0}^{+\infty}\delta_k < +\infty, \tag{21}$$

in particular, $\sum_{k=0}^{+\infty}\alpha_k\langle u^k, x^k - x^*\rangle < +\infty$.

Since $\rho_k^{-1} < \underline{\rho}^{-1}$ for all $k \in \mathbb{N}$ and the sequence $\{w^k\}$ is bounded, there exists $L > 0$ such that

$$\frac{\gamma_k}{\rho_k} = \max\{1, \rho_k^{-1}\|w^k\|\} \le L \quad \forall k \in \mathbb{N}$$

and with (11) we get

$$\frac{\beta_k}{L\rho_k} \le \alpha_k$$

it results

$$\sum_{k=0}^{+\infty}\frac{\beta_k}{L\rho_k}\langle u^k, x^k - x^*\rangle < +\infty.$$

We use the divergence of $\sum_{k=0}^{+\infty}\frac{\beta_k}{\rho_k}$ to conclude that

$$\limsup_{k\to+\infty}\langle u^k, x^* - x^k\rangle = 0.$$

Hence, there exists a subsequence $\{(x^{k_j}, u^{k_j}) \colon u^{k_j} \in T(x^{k_j})\}$ such that

$$\lim_{j\to+\infty}\langle u^{k_j}, x^* - x^{k_j}\rangle = \limsup_{k\to+\infty}\langle u^k, x^* - x^k\rangle = 0. \tag{22}$$

From the boundedness of $\{x^k\}$ and $\{u^k\}$, without loss of generality, there are $\overline{x}$, $\overline{u} \in \mathbb{R}^n$ with

$$\lim_{j\to+\infty} u^{k_j} = \overline{u} \quad \text{and} \quad \lim_{j\to+\infty} x^{k_j} = \overline{x}. \tag{23}$$

Since $\{x^k\} \subset C$ and the graph of the operator T is sequentially closed it results

$$\overline{x} \in C, \quad \overline{u} \in T(\overline{x}).$$

Therefore, $\langle\overline{u}, x^* - \overline{x}\rangle = 0$. From A2 and Proposition 1, we conclude that $\overline{x} \in S(T, C)$. By Theorem 1, $\{\|x^k - \overline{x}\|^2\}$ is convergent, hence

$$0 = \lim_{j\to+\infty}\|x^{k_j} - \overline{x}\| = \lim_{k\to+\infty}\|x^k - \overline{x}\|.$$

The proof is complete. $\square$

Remark 2 *When the algorithm (PMVI) becomes the usual projection method, that is, $w^k = -u^k$ for all $k \in \mathbb{N}$, the boundedness of the sequence $\{w^k\}$, in Theorem 2, is a consequence of the boundedness of the sequence $\{x^k\}$.*

4 Numerical tests

In this section we illustrate the behavior of the algorithm PMVI by considering a numerical test coded in SCILAB 5.2.2 on a 2GB RAM Intel Atom N450.

We consider the variational inequality problem that reformulates the Cournot oligopoly problem with shared constraints and nonlinear cost functions as described in Heusinger and Kanzow (2009). Following Harker (1984), the problem becomes

$$\begin{aligned}
&\textstyle\sum_{i=1}^{5} F_i(q_i - q_i^*) \geq 0 \ \ \forall \ \ q \in C,\\
&F_i(q) = f_i'(q) - p(\textstyle\sum_{j=1}^{5} q_j) - (q_i)p'(\textstyle\sum_{j=1}^{5} q_j),\\
&f_i(q_i) = c_i q_i + \frac{\beta_i}{\beta_i+1} K_i^{(-\frac{1}{\beta_i})} q_i^{(\frac{\beta_i+1}{\beta_i})},\\
&p(Q) = 5000^{\frac{1}{\eta}} Q^{-\frac{1}{\eta}},
\end{aligned}$$

with $\eta = 1.1$, $c = (10, 8, \cdots, 2)$, $K = (5, 5, \cdots, 5)$, $\beta = (1.2, 1.1, \cdots, 0.8)$, $C = \mathbb{R}_+^n$ and $q^* = (36.912, 41.842, 43.705, 42.665, 39.182)$.
For this problem, we use $w^k = -u^k$, $\beta_k = \frac{30}{k}$, $\rho_k = 1$ for all $k \in \mathbb{N}$.

In Table 1, we show the first three components of each iterate for sake of comparison of PMVI with the relaxation algorithm (RA) given in Heusinger and Kanzow (2009).

Table 1: Iterations of RA and PMVI

	RA			PMVI		
Iter.(k)	q_1^k	q_2^k	q_3^k	q_1^k	q_2^k	q_3^k
0	10.0000	10.0000	10.0000	10.0000	10.0000	10.0000
1	55.0181	56.1830	55.7512	22.2998	23.8567	23.4060
2	27.2067	33.0054	36.1694	27.9168	29.1315	30.2456
3	42.6043	46.7481	47.9981	31.5732	33.4380	35.0173
4	33.5762	38.8631	41.1775	34.3174	36.8889	38.8577
5	38.8777	43.5262	45.1860	36.5254	40.0134	42.2881
10	36.7970	41.6992	43.6040	36.8336	41.7204	43.6016
18	36.9306	41.8164	43.7051	36.9325	41.8181	43.7065
26	36.9324	41.8181	43.7065	–	–	–

5 Final remarks

In this paper, we have presented an inexact projection method where the step size is defined by a divergent series. We have showed the convergence of the generated sequence by considering that the operator T is pseudomonotone$_*$ and it has a sequentially closed graph.

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