A Proximal Scalarization Method with Logarithm and Quasi Distance to Multiobjective Programming

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ABSTRACT

Recently, Gregório and Oliveira developed a proximal point scalarization method (applied to multiobjective optimization problems) for an abstract strict scalar representation with a variant of the logarithmic-quadratic function of Auslender et al. as regularization. In this work we propose a variation of this method, taking into account the regularization with logarithm and quasi-distance, where we have lost important properties, such as the convexity. We show that the central trajectory of the scalarized problem is bounded and converges to a weak pareto solution of the multiobjective optimization problem.

KEYWORDS. Multiobjective Programming. Scalarization Method. Quasi Distance.

RESUMO

Recentemente, Gregório e Oliveira desenvolveram um método de escalarização proximal (Aplicado em problemas de Otimização Multiobjetivo) para uma representação escalar estrita abstrata com uma variante da função log-quadrática de Auslender et al. como regularização. Neste trabalho, propomos uma variação deste método considerando a regularização com logaritmo e quase distância, onde perdemos propriedades importantes, como a convexidade. Mostramos que a trajetória central do problema escalarizado é limitada e converge para uma solução pareto fraca do problema de otimização multiobjetivo.

Palavras Chave. Progamação Multiobjetivo. Método de escalarização. Quase distância.

1 Introduction

In this work we consider the unconstrained multiobjective optimization problem.

$$\min \ \{F(x); \quad x \in R^n\} \tag{1}$$

where $F = (F_1, F_2, ..., F_m)^T : R^n \to R^m$ is a convex mapping related to the lexcographic order generated by the cone R^m_+, i.e., for all $x, y \in R^n$ and $\lambda \in (0, 1)$,

$$F_i(\lambda x + (1 - \lambda)y) \leq \lambda F_i(x) + (1 - \lambda)F_i(y), \quad \forall i = 1, ..., m.$$

Moreover, we are going to demand that one of the objective functions must be coercive, i.e., there is $r \in \{1, ..., m\}$ such that $\lim_{\|x\|\to\infty} F_r(x) = \infty$. This class of problems (see, for example, Miettinen (1999)) is a particular case known as vetorial optimization (see, for example, in Luc (1989)).

The classic proximal point method to minimize a mono-objective convex function $f : R^n \to R$ generates a sequence $\{x^k\}$ via the iterative scheme: given a starting point $x^0 \in R^n$ we find

$$x^{k+1} \in \text{argmin} \ \{f(x) + \lambda_k \|x - x^k\|^2, \quad x \in R^n\},$$

where λ_k is a sequency of real positive numbers and $\|.\|^{1/2}$ is the usual norm. This method was originally introduced by Martinet (1970) and developed, and studied, by Rockafellar (1996). Literature related to the analysis and development of proximal point methods in a convex and non-convex includes Kaplan and Tichatschke (1998) and Kiwiel (1997). Moreno et al. (2011), developed a proximal method with a quasi distance as regularization, applied to non-convex and nonsmooth functions, and showed the importance of the behavior of this proximal point model to the economic area, specially to the habit formation in Decision and Making Sciences.

The proximal point methods were extended to vetorial optimization, check, for example, Miettinem and Mäkelä (1995), Gopfert et al. (2003), Bonnel et al. (2005). Gregório and Oliveira (2010), developed a proximal method, applied to multiobjective optimization problems, for a abstract strict scalar representation with a variant of the logarithmic-quadratic function of Auslender et al. (1999) as regularization.

Based on Gregório and Oliveira (2010), we have proposed a proximal method to a abstract strict scalar represetation considering as regularization a function involving a logarithm term and a quasi distance.

We show, into section 2, some concepts and results about the quasi distance and the subdifferential theory. Into section 3, we present some concepts and results of the optimization multiobjective general theory. Into section 4 we present our own method, where we assure the existence of the iterations, the stop criterion and the convergency. Finally, into section 5, we test our method showing some numerical examples using the Matlab.

2 Quasi Distance and Subdifferential Theory

In this section we define the quasi distance application, we present examples and some of its properties that are fundamental to the development of our work. We will also recall the concepts of Fréchet subdifferential and limiting-subdifferential with some of its properties.

2.1 Quasi Distance

Definition 1 (Moreno et al. (2011)) *Let X be a set. A mapping $q : X \times X \to R_+$ is called a quasi distance if for all $x, y, z \in X$,*

(i) $q(x, y) = q(y, x) = 0 \Longleftrightarrow x = y$ **(ii)** $q(x, z) \leq q(x, y) + q(y, z)$.

A quasi distance is not necessarily a convex function, continually differentiable and coercive (see Moreno et al. (2011) - Example 3.1 and Remark 3). Moreno et al. (2011) presented the following exemple of quasi distance.

Example 1 *For each $i = 1, ..., n$, we consider $c_i^-, c_i^+ > 0$ and $q_i : R \times R \to R_+$ defined by*

$$q_i(x_i, y_i) = \begin{cases} c_i^+(y_i - x_i) & if \quad y_i - x_i > 0 \\ c_i^-(x_i - y_i) & if \quad y_i - x_i \le 0 \end{cases}$$

is a quasi distance on R, therefore $q(x, y) = \sum_{i=1}^{n} q_i(x_i, y_i)$ is a quasi distance on R^n. On the other hand, for each $\bar{z} \in R^n$ we have

$$q(x, \bar{z}) = \sum_{i=1}^{n} q_i(x_i, \bar{z}_i) = \sum_{i=1}^{n} max\{c_i^+(\bar{z}_i - x_i), c_i^-(x_i - \bar{z}_i)\}, \qquad x \in R^n,$$

thus $q(., \bar{z})$ is a convex function. By the same reasoning, $q(\bar{z}, .)$ is convex.

Moreno et al. (2011) have taken into account the following condition about the quasi distance q: There are positive constants α and β such that

$$\alpha\|x - y\| \le q(x, y) \le \beta\|x - y\|, \qquad \forall x, y \in R^n \tag{2}$$

Proposition 1 (Moreno et al. (2011), Propositions 3.6 and 3.7) *Let $q : R^n \times R^n \to R_+$ be a quasi distance that verifies (2). Then for each $\bar{z} \in R^n$ the functions $q(\bar{z}, .)$ and $q(., \bar{z})$ are Lipschitz continuous and the functions $q^2(\bar{z}, .)$ and $q^2(., \bar{z})$ are locally Lipschitz continuous functions on R^n.*

Proposition 2 (Moreno et al. (2011), Remark 5) *Let $q : R^n \times R^n \to R_+$ be a quasi distance that verifies (2). Then for each $\bar{z} \in R^n$ the functions $q(\bar{z}, .)$, $q(., \bar{z})$, $q^2(\bar{z}, .)$ and $q^2(., \bar{z})$ are coercive.*

2.2 Subdifferential Theory

We recall now some concepts and results of Frechet subdifferential and limiting subdifferencial.

Definition 2 *Let $h : R^n \to R \cup \{\infty\}$ be a proper lower semicontinuous function and $x \in R^n$.*

1. *The Fréchet subdifferential of h at x, $\hat{\partial}h(x)$, is defined as follows*

$$\hat{\partial}h(x) := \begin{cases} \left\{x^* \in R^n : \liminf_{y \ne x, y \to x} \frac{h(y) - h(x) - \langle x^*, y - x\rangle}{\|x - y\|} \ge 0\right\}, & \text{if } x \in dom(h) \\ \emptyset, & \text{if } x \notin dom(h) \end{cases}$$

2. *The limiting-subdifferential of h at $x \in R^n$, $\partial h(x)$, is defined as follows*

$$\partial h(x) := \left\{x^* \in R^n : \exists x_n \to x, \quad h(x_n) \to h(x), \quad x_n^* \in \hat{\partial}h(x_n) \to x^*\right\}$$

Proposition 3 (Optimality condition - Rockafellar and Wets (1998), Theorem 10.1)

If a proper function $h : R^n \to R \cup \{+\infty\}$ has a local minimum at $\bar{x}$, then $0 \in \hat{\partial}h(\bar{x}), 0 \in \partial h(\bar{x})$.

Remark 1 *Be $C \subset R^n$. If a proper function $h : C \to R \cup \{\infty\}$ has a local minimun at $\bar{x} \in C$, then $0 \in \hat{\partial}(h + \delta_C)(\bar{x}), 0 \in \partial(h + \delta_C)(\bar{x})$, where δ_C is the indicator function of the set C, defined as $\delta_C(x) = 0$ if x belongs to C and $\delta_C(x) = \infty$ on the contrary.*

Proposition 4 (Rockafellar and Wets (1998), Exercise 10.10) *If f_1 is locally Lispschitz continuous at $\bar{x}$, f_2 is lower semicontinuous and proper with $f_2(\bar{x})$ finite, then*

$$\partial(f_1 + f_2)(\bar{x}) \subset \partial f_1(\bar{x}) + \partial f_2(\bar{x}).$$

Proposition 5 (Mordukhovich and Shao (1996), Theorem 7.1) *Let $f_i : R^n \to R, i = 1, 2,$ be Lipschitz continuous around $\bar{x}$. If $f_i \geq 0, i = 1, 2$. Then one has a product rule of the equatily form*

$$\partial(f_1.f_2)(\bar{x}) = \partial(f_2(\bar{x})f_1 + f_1(\bar{x})f_2)(\bar{x}).$$

Proposition 6 (Rockafellar and Wets (1998), Proposition 5.15) *A mapping $S : R^n \to P(R^m)$ is locally bounded if and only if $S(B)$ is bounded for every bounded set B.*

Proposition 7 (Rockafellar and Wets (1998), Theorem 9.13) *Suppose $h : R^n \to R \cup \{\pm\infty\}$ is locally lower semicontinuous at $\bar{x}$ with $h(\bar{x})$ finite. Then the following conditions are equivalent:*

(a) *h is locally Lipschitz continuous at $\bar{x}$,*

(b) *the mapping $\hat{\partial}h : x \mapsto \hat{\partial}h(x)$ is locally bounded at $\bar{x}$,*

(c) *the mapping $\partial h : x \mapsto \partial h(x)$ is locally bounded at $\bar{x}$.*

Moreover, when these conditions hold, $\partial h(\bar{x})$ is nonempty and compact.

3 Multiobjective programming - preliminary concepts

We are going to present only the concepts and results that are fundamental to the development of our work. For more details, see, for example, Miettinen (1999).

Definition 3 *We say that $a \in R^n$ is a* **local pareto solution** *to the problem (1) if there is a disc $B_\delta(a) \subset R^n$, with $\delta > 0$, such that there is no $x \in B_\delta(a)$ satisfying $F_i(x) \leq F_i(a)$ for all $i = 1, ..., m$ and $F_j(x) < F_j(a)$ for at least one index $j \in \{1, ..., m\}$.*

Definition 4 *$a \in R^n$ is known as* **weak local pareto solution** *if there is a disc $B_\delta(a) \subset R^n$, with $\delta > 0$, such that there is no $x \in B_\delta(a)$ satisfying $F_i(x) < F_i(a)$ for all $i = 1, ..., m$.*

In general, if a constrained or unconstrained multiobjective optimization problem is a convex problem, to say, if an objective function $F : R^n \to R^m$ is a convex function, then all (weak) local pareto solution is also a (weak) global pareto solution. This result is discussed in the 2.2.3 Theorem, in Miettinen (1999).

We will denote by $\text{argmin}\{F(x)|x \in R^n\}$ and $\text{argmin}_w\{F(x)|x \in R^n\}$ the local pareto solution set and the local weak pareto solution set to the problem (1). It is easy to see that $\text{argmin}\{F(x)|x \in R^n\} \subset \text{argmin}_w\{F(x)|x \in R^n\}$.

Definition 5 *A real valued function $f : R^n \longrightarrow R$ is said to be a* **strict scalar representation** *of a map $F = (F_1, ..., F_m) : R^n \longrightarrow R^m$ when given $x, \bar{x} \in R^n$*

$$F_i(x) \leq F_i(\bar{x}), \ \forall \ i = 1, ..., m \Longrightarrow f(x) \leq f(\bar{x})$$

and

$$F_i(x) < F_i(\bar{x}), \ \forall \ i = 1, ..., m \Longrightarrow f(x) < f(\bar{x}).$$

Futhermore, we say that f is a **weak scalar representation** *of F if*

$$F_i(x) < F_i(\bar{x}), \ \forall \ i = 1, ..., m \Longrightarrow f(x) < f(\bar{x}).$$

It is obvious that all strict scalar representations are weak scalar representations. The next result establishes an important relation between the sets $\text{argmin}\,\{f(x)|x \in R^n\}$ and $\text{argmin}_w\{F(x)|x \in R^n\}$. The Proof follows immediately from the Definition 5.

Proposition 8 *Let $f : R^n \longrightarrow R$ be a weak scalar representation of a map $F : R^n \longrightarrow R^m$ and $\text{argmin}\,\{f(x)|x \in R^n\}$ the local minimizer set of f. We have the inclusion*

$$\text{argmin}\,\{f(x)|x \in R^n\} \subset \text{argmin}_w\{F(x)|x \in R^n\}.$$

4 Proximal point scalarization method with logarithm and quasi distance - (LQDPS) Method

Gregório and Oliveira (2010) showed the existence of a function $f : R^n \times R^m_+ \longrightarrow R$ satisfying the following properties:

(P1) f is bounded below for any $\alpha \in R$, i.e, $f(x,z) \geq \alpha$ for every $(x,z) \in R^n \times R^m_+$;

(P2) f is convex in $R^n \times R^m_+$, i.e., given $(x_1,z_1),(x_2,z_2) \in R^n \times R^m_+$ and $\lambda \in (0,1)$

$$f(\lambda(x_1,z_1)+(1-\lambda)(x_2,z_2)) \leq \lambda f(x_1,z_1)+(1-\lambda)f(x_2,z_2);$$

(P3) f is a strict scalar representation of F, with respect to x, i.e.,

$$F_i(x) \leq F_i(y)\ \forall i=1,...,m\ \Rightarrow f(x,z) \leq f(y,z)$$

and

$$F_i(x) < F_i(y)\ \forall i=1,...,m\ \Rightarrow f(x,z) < f(y,z)$$

for every $x,y \in R^n$ and $z \in R^m_+$;

(P4) f is differentiable, with respect to z and

$$\frac{\partial}{\partial z} f(x,z) = h(x,z),$$

where $h(x,z) = (h_1(x,z),\cdots,h_m(x,z))^T$ is a continuous map from $R^n \times R^m$ to R^m_+, i.e, $h_i(x,z) \geq 0$ for all $i=1,\cdots,m$.

More precisely, they showed that the function $f : R^n \times R^m_+ \longrightarrow R$ such that

$$f(x,z) = \sum_{i=1}^{m} \exp(z_i + F_i(x)) \tag{3}$$

satisfies the properties $(P1)$ a $(P4)$. As another example, we present the following proposition:

Proposition 9 *Be $F=(F_1,F_2,...,F_m) : R^n \rightarrow R^m$ a convex application, then $f : R^n \times R^m_+ \rightarrow R$ such that $f(x,z) = \sum_{i=1}^{m}[z_i + h(F_i(x))]$ where $h(F_i(x)) = \begin{cases} \frac{1}{2-F_i(x)} & if \quad F_i(x) \leq 1 \\ (F_i(x))^2 & if \quad F_i(x) > 1 \end{cases}$ satisfies the properties (P1) to (P4).*

Proof. As $h : R \rightarrow R$ given by $h(x) = \begin{cases} \frac{1}{2-x} & if \quad x \leq 1 \\ x^2 & if \quad x > 1 \end{cases}$ is positive $(h > 0)$, convex and strictly increasing, it is easy to see that f satisfies the properties (P1) to (P4). ■

Notation: Be $y, \bar{y} \in R^m$, then $y \leq \bar{y} \iff y_i \leq \bar{y}_i\ \forall i=1,...,m$ and $y \ll \bar{y} \iff y_i < \bar{y}_i\ \forall i = 1,...,m$.

The Method (LQDPS):
Let $F : R^n \longrightarrow R^m$ be convex and $q : R^n \times R^n \rightarrow R_+$ a quasi distance application, satisfying (2).Given the initial points $x^0 \in R^n$, $z^0 \in R^m_{++}$ and sequences $\beta^k > 0, k = 0,1,\cdots$ and $0 < \mu_0 < \mu^k < \mu_1\ \forall k = 1,2,...$, the method (LQDPS) of proximal point scalarization with logarithm and quasi distance generates sequences $\{x^k\}_{k\in N} \subset R^n$ and $\{z^k\}_{k\in N} \subset R^m_{++}$ with the iterates x^{k+1} and z^{k+1} defined as the solution of the $(LQDPS)$ problem

$$\min \varphi^k(x,z) = f(x,z) + \beta^k \langle \frac{z}{z^k} - \log\frac{z}{z^k} - e, e\rangle + \frac{\mu^k}{2} q^2(x,x^k), \tag{4}$$
$$x \in \Omega^k, z \in R^m_{++},$$

where $f : R^n \times R^m_+ \longrightarrow R$ verifies the properties (P1) to (P4), $\frac{z}{z^k}$ and $\log\frac{z}{z^k}$ which are the vectors whose ith components are given by $\frac{z_i}{z_i^k}$ and $\log\frac{z_i}{z_i^k}$, respectively, $e \in R^m$ is the vector with all components equal to 1 and $\Omega^k = \{x \in R^n | F(x) \leq F(x^k)\}$.

4.1 Well-posedness

The function $\varphi^k : R^n \times R^m_{++} \longrightarrow R$ in (4), was considered by Gregório and Oliveira (2010) having as regularization a variant of the logarithm-quadratic function of Auslender et al. (1999) and, in this case, due to the strict convexity of the fuction φ^k, they have showed that the iteractions of the method are unique and interior the restrictions. As the quasi distance is not necessarily a convex function, we will not assure the uniqueness of the iteractions and we will not assure also that the iteractions x^{k+1} are interior the restrictions Ω^k. Therefore, we will have to act differently to assure the good definition of the sequencies and their respective characterizations. It is easy to prove that:

Lemma 1 *Let $F : R^n \longrightarrow R^m$ be a convex map such that there exists $r \in \{1, ..., m\}$ satisfying $\lim_{\|x\|\to\infty} F_r(x) = \infty$. Then, Ω^k is a convex and compact set. Particularly, $\Omega^k \times R^m_+$ is a convex and closed set.*

Proof. Suppose, for contradiction that $\Omega^0 = \{x \in R^n | F(x) \leq F(x^0)\}$ is unbounded. Then there is $\{x_n\}_{n\in N} \subset \Omega^0$ such that $\|x_n\| \to \infty$ when $n \to \infty$. As $\{x_n\}_{n\in N} \subset \Omega^0$ we have $F(x_n) \leq F(x^0)$ $\forall n \in N$, and then, $F_i(x_n) \leq F_i(x^0)$, $\forall\ i = 1, ..., m$ and $n \in N$. Therefore, in particular, $F_r(x_n) \leq F_r(x^0)$ $\forall n \in N$. Since F_r is coercive and $\|x_n\| \to \infty$ when $n \to \infty$ we "$\infty \leq F_r(x^0) < \infty$", that is a contradiction. So Ω^0 is limited. As $\Omega^{k+1} \subseteq \Omega^k$, $k \geq 0$, it follows that $\Omega^k \subseteq \Omega^0$, $k \geq 1$ and therefore Ω^k is limited $\forall k \geq 0$. The convexity of F implies its continuity and the convexity of Ω^k, $\forall k$. It is followed from the continuity of F that Ω^k, $\forall k$ is closed. Therefore, Ω^k $\forall k$ is a compact convex set. ■

Lemma 2 *The fuction $H : R^m_{++} \to R$ such that*

$$H(z) = \left\langle \tfrac{z}{z^k} - \log \tfrac{z}{z^k} - e, e \right\rangle = \|\frac{z}{z^k} - \log\ \frac{z}{z^k} - e\|_1$$

where $\| \bullet \|_1$ is the 1–norm on R^m defined by $\|z\|_1 = \sum_{i=1}^{m} |z_i|$ is strictly convex,non negative and coercive.

Proof. See Gregório and Oliveira (2010), demonstration of Lema 1. ■

As long as $H : R^m_{++} \to R$ is coercive, we can consider $H : R^m_+ \to R \cup \{\infty\}$ and therefore, $\varphi^k : R^n \times R^m_+ \longrightarrow R \cup \{\infty\}$.

Proposition 10 (Well-posedness) *Let $F : R^n \longrightarrow R^m$ be a convex map such that there exists $r \in \{1, ..., m\}$ satisfying $\lim_{\|x\|\to\infty} F_r(x) = \infty$, $q : R^n \times R^n \to R_+$ a quasi distance map satisfying (2) and $f : R^n \times R^m_+ \longrightarrow R$ be a function verifying the properties (P1) to (P4). Then, for every $k \in N$, there is one solution (x^{k+1}, z^{k+1}) for the $(LQDPS)$ problem.*

Proof. The function $\varphi^k : \Omega^k \times R^m_{++} \to R$ is coercive. In fact, for (P1) we have:

$$\begin{aligned} \varphi^k(x, z) &= f(x, z) + \beta^k \left\langle \frac{z}{z^k} - \log \frac{z}{z^k} - e, e \right\rangle + \frac{\mu^k}{2} q^2(x, x^k) \\ &\geq \alpha + \beta^k(\|\frac{z}{z^k} - \log\ \frac{z}{z^k} - e\|_1) + \frac{\mu^k}{2} q^2(x, x^k). \end{aligned} \tag{5}$$

Let us define $\|(x, z)\| = \|x\| + \|z\|$ and suppose that $\|(x, z)\| \to \infty$. This is the same as $\|x\| \to \infty$ or $\|z\| \to \infty$. As Ω^k is compact (see lema 1) and the function $\|\frac{z}{z^k} - \log\ \frac{z}{z^k} - e\|_1$ is coercive in R^m_{++} (see lema 2), it follows from (5) that φ^k is coercive in $\Omega^k \times R^m_{++}$.
The function $\varphi^k : R^n \times R^m_{++} \to R$ is continuous in $R^n \times R^m_{++}$. In fact: (P2) implies f continoues in $R^n \times R^m_{++}$. The lema 2 implies $H(z) = \left\langle \frac{z}{z^k} - \log \frac{z}{z^k} - e, e \right\rangle$ continuous in R^m_{++}. As a consequence of proposition 1, $q^2(., x^k) : R^n \to R$ is a continuous application in R^n.

Threfore, the function $\varphi^k : R^n \times R^m_{++} \to R \cup \{+\infty\}$ is continuous in $R^n \times R^m_{++}$.
As $\varphi^k : \Omega^k \times R^m_{++} \to R$ is a continuous, coercive and proper in $\Omega^k \times R^m_{++}$, we have that the set $\operatorname{argmin}\{\varphi^k(x,z)/(x,z) \in \Omega^k \times R^m_{++}\}$ is not empty, i.e., to every k, there is a solution $\left(x^{k+1}, z^{k+1}\right)$ to the problem (LQDPS). ■

Definition 6 *Let $C \subset R^n$ be a convex set and $\bar{x} \in C$. The normal cone (Cone of normal directions) at the pointo $\bar{x}$ related to the set C is given by*

$$N_C(\bar{x}) = \{v \in R^n \quad / \quad \langle v, x - \bar{x}\rangle \leq 0 \quad \forall x \in C\}.$$

Corollary 1 (Characterization)
The solutions $\left(x^{k+1}, z^{k+1}\right)$ of the problems LQDPS are characterized by:

(i) There are $\xi^{k+1} \in \partial f(., z^{k+1})(x^{k+1})$, $\zeta^{k+1} \in \partial(q(., x^k))(x^{k+1})$ and $v^{k+1} \in N_{\Omega^k}(x^{k+1})$ such that

$$\xi^{k+1} = -\mu^k q(x^{k+1}, x^k)\zeta^{k+1} - v^{k+1} \tag{6}$$

and

(ii)

$$\frac{1}{z_i^{k+1}} - \frac{1}{z_i^k} = \frac{h_i\left(x^{k+1}, z^{k+1}\right)}{\beta^k}, \quad i = 1, \cdots m. \tag{7}$$

$$x^{k+1} \in \Omega^k, z^{k+1} \in R^m_{++}$$

Proof.
By observation 1 we have

$$0 \in \partial\left(f(., z^{k+1}) + \beta^k\left\langle \frac{z^{k+1}}{z^k} - \log\frac{z^{k+1}}{z^k} - e, e\right\rangle + \frac{\mu^k}{2}q^2(., x^k) + \delta_{\Omega^k}\right)(x^{k+1}). \tag{8}$$

For (P2), $f(., z^{k+1}) + \beta\left\langle \frac{z^{k+1}}{z^k} - \log\frac{z^{k+1}}{z^k} - e, e\right\rangle$ is continuous in x^{k+1}, from the proposition 1, $\frac{\mu^k}{2}q^2(., x^k)$ is locally lipschitz in x^{k+1}, the convexity of Ω^k implies in the convexity of δ_{Ω^k} and therefore that δ_{Ω^k} is locally lipschitz, then, using the proposition 4 em (8) and remembering that

$$\beta\left\langle \frac{z^{k+1}}{z^k} - \log\frac{z^{k+1}}{z^k} - e, e\right\rangle$$

is constant into relation to Ω^k, we obtain

$$0 \in \partial\left(f(., z^{k+1})\right)(x^{k+1}) + \partial\left(\frac{\mu^k}{2}q^2(., x^k)\right)(x^{k+1}) + \partial\left(\delta_{\Omega^k}\right)(x^{k+1}). \tag{9}$$

As Ω^k is closed and convex, it follows $\partial\left(\delta_{\Omega^k}(.)\right)(x^{k+1}) = N_{\Omega^k}(x^{k+1})$, where $N_{\Omega^k}(x^{k+1})$ denotes the normal cone in the point x^{k+1} in relation to the set Ω^k (see def. 6). From the propositon 1, $q(., x^k)$ is Lipschitz continuous in R^n. Therefore, taking $f_1 = f_2 = q$ in the proposition 5, we have of (9) that

$$0 \in \partial\left(f(., z^{k+1})\right)(x^{k+1}) + \mu^k q(x^{k+1}, x^k)\partial\left(q(., x^k)\right)(x^{k+1}) + N_{\Omega^k}(x^{k+1}),$$

i.e., there are $\xi^{k+1} \in \partial f(., z^{k+1})(x^{k+1})$, $\zeta^{k+1} \in \partial(q(., x^k))(x^{k+1})$ and $v^{k+1} \in N_{\Omega^k}(x^{k+1})$ such that

$$\xi^{k+1} = -\mu^k q(x^{k+1}, x^k)\zeta^{k+1} - v^{k+1}.$$

To end the demonstration, we observe (see Gregório and Oliveira (2010), Lemma 1) that

$$\frac{1}{z_i^{k+1}} - \frac{1}{z_i^k} = \frac{h_i\left(x^{k+1}, z^{k+1}\right)}{\beta^k}, \quad i = 1, \cdots m.$$

$$x^{k+1} \in \Omega^k, z^{k+1} \in R^m_{++}$$

■

4.2 STOP CRITERION

As Gregório and Oliveira (2010), we are going to stablish the same stopping rule that was used by Bonnel et al. (2005).

Proposition 11 (Stop criterion) *Let $\{(x^k, z^k)\}_{k\in N}$ be the sequence generated by the $(LQDPS)$ method. If $(x^{k+1}, z^{k+1}) = (x^k, z^k)$ for any integer k then x^k is a weak pareto solution for the unconstrained multiobjective optimization problem (1).*

Proof. Now, suppose that the stopping rule is verified in the kth iteration. By contradiction, admit that x^k is not a weak pareto solution. Then, there is $\bar{x} \in R^n$ such that $F(\bar{x}) \ll F(x^k)$. By (P3) we have

$$f(\bar{x}, z^k) < f(x^k, z^k).$$

This implies that exists $\alpha > 0$ such that $f(\bar{x}, z^k) = f(x^k, z^k) - \alpha$. Defined $x_\lambda = \lambda x^k + (1 - \lambda)\bar{x}, \lambda \in (0, 1)$. We have that

$$(x_\lambda, z^k) = \lambda(x^k, z^k) + (1 - \lambda)(\bar{x}, z^k).$$

Since (x^{k+1}, z^{k+1}) solves the $(LQDPS)$ problem, $(x^{k+1}, z^{k+1}) = (x^k, z^k)$, $q^2(x^k, x^k) = 0$ and $x_\lambda \in \Omega^k$, $\forall \lambda \in (0, 1)$, we obtain,

$$f(x^k, z^k) \leq f(x_\lambda, z^k) + \tfrac{\mu^k}{2} q^2(x_\lambda, x^k), \ \forall \ \lambda \in (0, 1).$$

Of (2), we have,

$$f(x^k, z^k) \leq f(x_\lambda, z^k) + \frac{\mu^k}{2}\beta^2 \|x_\lambda - x^k\|^2, \ \forall \ \lambda \in (0, 1). \tag{10}$$

As $x_\lambda - x^k = (1 - \lambda)(\bar{x} - x^k)$, of (10) we obtain

$$f(x^k, z^k) \leq f(x_\lambda, z^k) + \frac{\mu^k}{2}\beta^2 (1-\lambda)^2 \|\bar{x} - x^k\|^2, \ \forall \ \lambda \in (0, 1). \tag{11}$$

On the other hand, the convexity of f implies that

$$\begin{aligned} f(x_\lambda, z^k) &\leq \lambda f(x^k, z^k) + (1-\lambda) f(\bar{x}, z^k) \\ &= \lambda f(x^k, z^k) + (1-\lambda)(f(x^k, z^k) - \alpha) \\ &= f(x^k, z^k) - (1-\lambda)\alpha. \end{aligned} \tag{12}$$

From (11) and (12), $f(x^k, z^k) \leq f(x^k, z^k) - (1-\lambda)\alpha + \frac{\mu^k}{2}\beta^2(1-\lambda)^2\|\bar{x} - x^k\|^2$. So

$$\alpha \leq (1-\lambda)\tfrac{\mu^k}{2}\beta^2 \|\bar{x} - x^k\|^2, \ \forall \lambda \in (0, 1).$$

Hence, $\alpha \leq \lim\limits_{\lambda \to 1^-} (1-\lambda)\dfrac{\mu^k}{2}\beta^2\|\bar{x} - x^k\|^2$, and therefore, $\alpha \leq 0$, that is a contradiction. So x^k is a weak pareto solution for the unconstrained multiobjective optimization problem (1). ■

4.3 CONVERGENCE

Based on Fliege and Svaiter (2000), Gregório and Oliveira (2010) supposed that Ω^0 is limited and stablished the convergency of the proximal scalarization method log-quadratic. In this work, we assume that one of the objective functions is coercive, that has as consequence the limitation of Ω^0, see lema 1.

Proposition 12 *Let $\{(x^k, z^k)\}_{k\in N}$ be a sequence generated by* **Method (LQDPS)**. *Then* (i) *$\{x^k\}_{k\in N}$ is é bounded;* (ii) *$\{z^k\}_{k\in N}$ is convergent;* (iii) *$\{f(x^k, z^k)\}_{k\in N}$ is convergent.*

Proof. **(i)** Since $\Omega^k \supseteq \Omega^{k+1}, k = 0, 1, ...,$ we have $x^k \in \Omega^{k-1} \subseteq \Omega^0 \quad \forall k \geq 1$. As Ω^0 is limited, it follows that $\{x^k\}$ is limited.
(ii) Since $h_i(x, z) \geq 0$, $\beta^k > 0$ and $\{z_i^k\}_{k \in N}$ is bounded bellow, the Equation (7) implies $\{z^k\}_{k \in N}$ is convergent (see, [5], proof of theorem 1).
(iii) $\varphi^k(x^{k+1}, z^{k+1}) \leq \varphi^k(x^k, z^k), \forall k \in N$, i.e, to every $k \in N$,

$$f(x^{k+1}, z^{k+1}) + \beta^k \left\langle \frac{z^{k+1}}{z^k} - \log \frac{z^{k+1}}{z^k} - e, e \right\rangle + \frac{\mu^k}{2} q^2(x^{k+1}, x^k) \leq f(x^k, z^k). \tag{13}$$

As $\beta^k \left\langle \frac{z^{k+1}}{z^k} - \log \frac{z^{k+1}}{z^k} - e, e \right\rangle + \frac{\mu^k}{2} q^2(x^{k+1}, x^k) \geq 0 \ \forall k \in N$, we have,

$$f(x^{k+1}, z^{k+1}) \leq f(x^k, z^k) \ \forall k \in N,$$

i.e., $\{f(x^k, z^k)\}_{k \in N}$ is a noincreasing sequence. For (P1), $\{f(x^k, z^k)\}$ is bounded lower, and therefore convergent. ■

Proposition 13 *Let* $\{x^k\}_{k \in N}$ *be a sequence generated by* **Method (LQDPS)**. *Then*

(i) $\sum_{k=0}^{\infty} q^2(x^{k+1}, x^k) < \infty$. *In particular* $\lim_{k \to \infty} q^2(x^{k+1}, x^k) = 0$.

(ii) $\lim_{k \to \infty} \|x^k - x^{k+1}\| = 0$.

Proof. **(i)** As $\beta^k \left\langle \frac{z^{k+1}}{z^k} - \log \frac{z^{k+1}}{z^k} - e, e \right\rangle \geq 0$, of (13) we have:

$$f(x^{k+1}, z^{k+1}) + \frac{\mu^k}{2} q^2(x^{k+1}, x^k) \leq f(x^k, z^k), \ \forall k \in N.$$

Hence,
$$\begin{aligned} q^2(x^{k+1}, x^k) &\leq \frac{2}{\mu^k} \left(f(x^k, z^k) - f(x^{k+1}, z^{k+1}) \right), \forall k \in N \\ &\leq \frac{2}{\mu_0} \left(f(x^k, z^k) - f(x^{k+1}, z^{k+1}) \right), \forall k \in N. \end{aligned}$$

Therefore, as long as $\{f(x^k, z^k)\}_{k \in N}$ is noincreasing and convergent,

$$\sum_{k=0}^{n} q^2(x^{k+1}, x^k) \leq \frac{2}{\mu_0} \left(f(x^0, z^0) - \lim_{k \to \infty} f(x^{k+1}, z^{k+1}) \right) < \infty \quad \forall n \in N$$

(ii) (2) implies $\alpha^2 \|x^k - x^{k+1}\|^2 \leq q^2(x^{k+1}, x^k), \ \forall \ k \in N$. So of **(i)**, $\lim_{k \to \infty} \|x^k - x^{k+1}\| = 0$. ■

Proposition 14 *If* $\{x^k\}_{k \in N}$ *is bounded, then the set* $\partial \left(q(., x^k) \right) (x^{k+1})$ *is bounded to every* $k \in N$.

Proof. It follows from the propositions (6) and (7), see Moreno et al. (2011), Lema 5.1. ■

Now, we can prove the convergence of our method if the stopping rule never applies.

Theorem 1 (convergence) *Let* $F : R^n \longrightarrow R^m$ *be a convex map such that* $\lim_{\|x\| \to \infty} F_r(x) = \infty$ *for some* $r \in \{1, ..., m\}$, $f : R^n \times R^m_+ \longrightarrow R$ *be a function verifying the properties (P1) to (P4) and* $q : R^n \times R^n \to R_+$ *be a function quasi distance satisfazendo (2) . If* $\{\mu^k\}_{k \in N}$ *and* $\{\beta^k\}_{k \in N}$ *are sequences of real positive numbers, with* $0 < \mu_0 < \mu^k < \mu_1, \forall \ k \in N$, *then the sequence* $\{(x^k, z^k)\}_{k \in N}$ *generated by the proximal point scalarization Method with logarithm and quasi distance is bounded and each cluster point of* $\{x^k\}_{k \in N}$ *is a weak pareto solution for the unconstrained multiobjective optimization problem (1).*

Proof. From the proposition 12, there are $x^* \in R^n$, $z^* \in R^m_+$ and $\{x^{k_j}\}_{j\in N}$ subsequency of $\{x^k\}_{k\in N}$ such that $\lim_{j\to\infty} x^{k_j} = x^*$ and $\lim_{k\to\infty} z^k = z^*$. By (P2) f is continuous in $R^n \times R^m_{++}$, so $\lim_{k\to\infty} f(x^{k_j}, z^{k_j}) = f(x^*, z^*) = \inf_{k\in N}\{f(x^k, z^k)\}$. From corollary 1(i), there are $\zeta^{k+1} \in \partial(q(., x^k))(x^{k+1})$ and $v^{k+1} \in N_{\Omega^k}(x^{k+1})$ such that

$$-\mu^k q(x^{k+1}, x^k)\zeta^{k+1} - v^{k+1} \in \partial f(., z^{k+1})(x^{k+1}).$$

Hence, from subgradient inequality to the convex function $f(., z^{k+1})$ we have: $\forall x \in R^n$,

$$\begin{aligned} f(x, z^{k_j+1}) &\geq f(x^{k_j+1}, z^{k_j+1}) - \mu^{k_j} q(x^{k_j+1}, x^{k_j}) < \zeta^{k_j+1}, x - x^{k_j+1} > \\ &- < v^{k_j+1}, x - x^{k_j+1} > \end{aligned} \tag{14}$$

As $v^{k_j+1} \in N_{\Omega^{k_j}}(x^{k_j+1})$ we have $- < v^{k_j+1}, x - x^{k_j+1} > \geq 0 \quad \forall x \in \Omega^{k_j}$ (See definition 6). Therefore, in particular, of (14) we have: $\forall x \in \Omega^{k_j}$,

$$f(x, z^{k_j+1}) \geq f(x^{k_j+1}, z^{k_j+1}) - \mu^{k_j} q(x^{k_j+1}, x^{k_j}) < \zeta^{k_j+1}, x - x^{k_j+1} > \tag{15}$$

From the propositions 13 e 14, $\lim_{k\to\infty} \|x^{k_j} - x^{k_j+1}\| = 0$ and $\|\zeta^{k_j+1}\| \leq M$ respectivelly. As $0 < \mu_0 < \mu^k < \mu_1, \forall\ k \in N$, using (2) and inequality of Cauchy-Swartz we conclude that $| \mu^{k_j} q(x^{k_j+1}, x^{k_j}) < \zeta^{k_j+1}, x - x^{k_j+1} >| \to 0$ when $j \to \infty$. Therefore from (15),

$$f(x, z^*) \geq f(x^*, z^*),\ \forall\ x \in \Omega^{k_j}. \tag{16}$$

We are going to show now that $x^* \in \text{argmin}_w\{F(x)/x \in R^n\}$. Suppose, by contradiction, that there is $\bar{x} \in R^n$ such that $F(\bar{x}) \ll F(x^*)$. As $z^* \in R^m_+$, for (P3),

$$f(\bar{x}, z^*) < f(x^*, z^*). \tag{17}$$

As $\Omega^{k+1} \subseteq \Omega^k$, $\forall k \geq 0$ and $x^{k_j} \in \Omega^{k_j-1}$, $\forall j$ with $x^{k_j} \to x^*$; $j \to \infty$ we have that $x^* \in \Omega^{k_j}$, i.e., $F(x^*) \leq F(x^{k_j})$. Hence $F(\bar{x}) \ll F(x^{k_j})$, i.e, $\bar{x} \in \Omega^{k_j}$, which contradicts (16) and (17). ∎

5 Numerical examples

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In this section we are going to implement the LQDPS method given into section 4. All numerical experiences were done in an intel(R) Core(TM) 2 Duo with Windows 7 installed and the source code is written in Matlab 7.9.0. We have tested our method taking into account three multiobjective test functions that were presented by Li and Zhang (2009), that is, we have taken into account the following functions:

(a) ([8], function F1, pg. 287): $F_a = (F_a^1, F_a^2) : R^3 \to R^2$ given for $F_a^1 = x_1 + 2(x_3 - x_1^2)^2$, $F_a^2 = 1 - \sqrt{x_1} + 2(x_2 - x_1^{0,5})^2$ and $x_i \in [0, 1]$, $i = 1,\ 2,\ 3$ which set of all Pareto optimal points (PS) is given for $x_2 = x_1^{0,5}$ and $x_3 = x_1^2$, $x_1 \in [0, 1]$.
(b) ([8], function F4, pg. 287): $F_b = (F_b^1, F_b^2) : R^3 \to R^2$ given for $F_b^1 = x_1 + 2(x_3 - 0,8x_1\cos((6\pi x_1 + \pi)/3))^2$, $F_b^2 = 1 - \sqrt{x_1} + 2(x_2 - 0,8x_1\sin(6\pi x_1 + 2\pi/3))^2$ and $(x_1, x_2, x_3) \in [0, 1] \times [-1, 1] \times [-1, 1]$ with the set (PS) given by $x_2 = 0,8x_1\sin(6\pi x_1 + 2\pi/3)$ and $x_3 = 0.8x_1\cos((6\pi x_1 + \pi)/3)$, $x_1 \in [0, 1]$.
(c) ([8], function F6, pg. 287): $F_c = (F_c^1, F_c^2, F_c^3) : R^3 \to R^3$ given for: $F_c^1 = \cos(0,5x_1\pi)\cos(0,5x_2\pi)$, $F_c^2 = \cos(0,5x_1\pi)\sin(0,5x_2\pi)$, $F_c^3 = \sin(0,5x_1\pi) + 2(x_3 - 2x_2\sin(2\pi x_1 + \pi))^2$ and $(x_1, x_2, x_3) \in [0, 1] \times [0, 1] \times [-2, 2]$ with the set (PS) given by $x_3 = 2x_2\sin(2\pi x_1 + \pi)$, $(x_1, x_2) \in [0, 1] \times [0, 1]$.

In the tables below we denote for tol, the tolerance related to the stop criterion ($\|(x^k, z^k) - (x^{k+1}, z^{k+1})\|_\infty \leq tol$); μ_k, β_k are the parameters of the (LQDPS) method; k_i^*, $i = 1, 2$ the iteractions number of the algorithm using the scalarization function $f_i : R^n \times R^m_+ \to R$, $i = 1, 2$ where f_1 is given through the proposition 9 and f_2 is given by (3); $\|x^*_{k_i^*} - x^*\|_\infty$, the distance, related to the infinite norm, of the approximated solution related to f_i and the exact solution, i.e., the mistake commited with the scalarization function f_i. The maximum number of iterations is 100. In all tests we are going to consider the quasi distance application contained in the example 1.

Example 2 *In this example, we are going to consider the Multiobjective function $F_a : R^3 \to R^2$ given above, and the initial iterations $x_0 = (0.5, 0.5, 0.5) \in R^3$ and $z_0 = (1, 1) \in R^2_{++}$. The numeric results are presented in the table below.*

No.	tol	μ_k	β_k	k_1^*	$\|x^*_{k_1^*} - x^*\|_\infty$	k_2^*	$\|x^*_{k_2^*} - x^*\|_\infty$
1	10^{-2}	$1+1/k$	$1+1/k$	9	5.545339e-003	10	5.118762e-002
2	10^{-3}	$1+1/k$	$1+1/k$	28	6.247045e-009	23	6.979995e-003
3	10^{-4}	$1+1/k$	$1+1/k$	87	7.960987e-009	62	8.279647e-009
4	10^{-2}	$1+1/k$	k	7	1.701151e-002	9	5.726488e-002
5	10^{-3}	$1+1/k$	k	28	7.351215e-009	24	6.281176e-003
6	10^{-4}	$1+1/k$	k	100	3.576296e-009	41	8.260435e-009
7	10^{-2}	$2-1/k$	$1/k$	7	2.273775e-002	8	9.389888e-002
8	10^{-3}	$2-1/k$	$1/k$	15	2.790977e-003	32	1.040779e-002
9	10^{-4}	$2-1/k$	$1/k$	28	1.071720e-008	100	9.213105e-009
10	10^{-2}	$2-1/k$	k	7	1.674806e-002	8	9.413130e-002
11	10^{-3}	$2-1/k$	k	27	8.168611e-009	32	1.039107e-002
12	10^{-4}	$2-1/k$	k	100	8.096220e-009	65	7.790086e-009
13	10^{-2}	1	1	8	6.966285e-003	9	5.000950e-002
14	10^{-3}	1	1	26	1.906054e-009	23	6.138829e-003
15	10^{-4}	1	1	83	8.254353e-009	39	1.546241e-005

Example 3 *In this example we consider the Multiobjective function $F_b : R^3 \to R^2$ given above, and the initial iterations $x_0 = (0.5, 0.5, 0.5) \in R^3$ and $z_0 = (1, 1) \in R^2_{++}$. the numeric results are presented in the table below.*

No.	tol	μ_k	β_k	k_1^*	$\|x^*_{k_1^*} - x^*\|$	k_2^*	$\|x^*_{k_2^*} - x^*\|_\infty$
1	10^{-2}	$1+1/k$	$1+1/k$	10	4.419117e-003	10	3.800596e-002
2	10^{-3}	$1+1/k$	$1+1/k$	29	7.617346e-009	20	5.872760e-003
3	10^{-4}	$1+1/k$	$1+1/k$	92	7.831306e-009	100	8.102059e-009
4	10^{-2}	$1+1/k$	k	7	1.423293e-002	10	3.771631e-002
5	10^{-3}	$1+1/k$	k	20	4.560126e-009	21	5.533943e-003
6	10^{-4}	$1+1/k$	k	98	6.872232e-009	38	1.000619e-007
7	10^{-2}	$2-1/k$	$1/k$	7	2.265857e-002	9	6.038495e-002
8	10^{-3}	$2-1/k$	$1/k$	15	3.304754e-003	25	8.176106e-003
9	10^{-4}	$2-1/k$	$1/k$	28	7.814512e-009	100	7.497307e-009
10	10^{-2}	$2-1/k$	k	7	1.251182e-002	7	6.525365e-002
11	10^{-3}	$2-1/k$	k	30	8.735791e-009	23	8.117802e-003
12	10^{-4}	$2-1/k$	k	100	5.561728e-009	52	7.547563e-009
13	10^{-2}	1	1	8	5.099261e-003	10	4.231940e-002
14	10^{-3}	1	1	27	5.045036e-009	22	5.051759e-003
15	10^{-4}	1	1	88	8.499673e-009	82	9.235203e-010

Example 4 *In this example we consider the Multiobjective function $F_c : R^3 \to R^3$ given above,and the initial iterations $x_0 = (0.5, 0.5, 0.5) \in R^3$ and $z_0 = (1, 1, 1) \in R^3_{++}$. The numeric results are presented in the table below.*

No.	tol	μ_k	β_k	k_1^*	$\|x^*_{k_1^*} - x^*\|$	k_2^*	$\|x^*_{k_2^*} - x^*\|_\infty$
1	10^{-2}	$1+1/k$	$1+1/k$	10	1.066481e-002	18	5.068830e-002
2	10^{-3}	$1+1/k$	$1+1/k$	31	1.698174e-008	33	5.315241e-003
3	10^{-4}	$1+1/k$	$1+1/k$	100	5.432795e-009	100	1.130028e-008
4	10^{-2}	$1+1/k$	k	10	9.733382e-003	19	5.908485e-002
5	10^{-3}	$1+1/k$	k	28	4.176586e-010	34	2.307318e-007
6	10^{-4}	$1+1/k$	k	100	7.086278e-011	35	7.450585e-009
7	10^{-2}	$2-1/k$	$1/k$	11	2.653977e-002	20	9.899806e-002
8	10^{-3}	$2-1/k$	$1/k$	18	2.046561e-007	47	1.059293e-002
9	10^{-4}	$2-1/k$	$1/k$	33	1.161832e-008	100	9.253656e-009
10	10^{-2}	$2-1/k$	k	11	1.835990e-002	22	8.862843e-002
11	10^{-3}	$2-1/k$	k	28	5.441347e-010	48	1.047200e-002
12	10^{-4}	$2-1/k$	k	100	9.476497e-010	75	2.793537e-009
13	10^{-2}	1	1	9	8.326796e-003	17	5.182457e-002
14	10^{-3}	1	1	29	1.343175e-008	32	4.799238e-003
15	10^{-4}	1	1	96	6.882171e-009	100	3.961009e-009

6 Conclusions

We propose a condition in one of the objective function that has as a consequence the limitation of Ω^0 and we propose another example of a function that satisfies the properties (P1) to (P4). As a variation of the Logarithm-Quadratic proximal scalarization method of Gregório and Oliveira (2010), we replaced the quadratic term with the quasi distance, where we have lost important properties as, for example, the convexity. However, acting in a different way, we proved the convergence of the method.

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