ACCENTUATING THE RANK POSITIONS IN AN AGREEMENT INDEX WITH REFERENCE TO A CONSENSUS ORDER

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ABSTRACT

In this paper, we propose an index that measures the agreement level between an individual opinion and a collective opinion when both are expressed by rankings of a set of alternatives. This index constitutes an interesting weighted version of the well-known Kendall's ranks correlation index. The originality of the proposed index arises from the fact that it accounts for the position weights of the alternatives in an individual order to quantify the agreement level of the individual order with respect to a collective temporary order. The proposed index is then used to compute, within an interactive and iterative procedure for reaching a consensus order, the agreement level of all of the group's members with respect to a collective order.

KEYWORDS. Group multicriteria decision aid, Consensus ranking, Kendall's distance

ADM - Multicriteria Decision Support

1. Introduction

In the context of the group multi-criteria ranking problem, the practice of reaching a consensus is a necessity because the desired goal is to achieve a general consensus of the ranking of alternatives. This practice can be supported by indices for computing the consensus or agreement level of an individual opinion with respect to the group opinion. Usually, the group opinion is derived from the aggregation of the individual opinions by using an aggregation procedure.

According to the above considerations, the purpose of this paper is to propose an index that measures the agreement level between an individual opinion and a collective opinion when both are expressed by rankings of a set of alternatives. The freshness of the proposed index results from the following facts: i) The proposed index is based on the Kendall metric. One of the reasons that metrics such as the Kendall metric remain prominent, even when their shortfalls have been widely recognized, is their inherent simplicity. It is natural to count the total number of inversions. However, as the metric becomes richer, sometimes this simplicity is lost; in this paper, we propose a new measure that conserves the inherent simplicity of the Kendall metric. ii) The proposed index accounts for the position weights. Intuitively, an inversion of an alternative in a high position should be more significant than an inversion of an alternative in a low position; inversions at the top of the ranking are costlier than inversions at the tail of the ranking. iii) We show that the proposed measure collapses to the classical variant when all of the position weights are set to 1.

This agreement index is very practical when the group's members use an interactive procedure for reaching a consensus order. In fact, in many decision-making situations of loosely coupled collaborative groups, the collective order obtained from the aggregation of the individual orders may not be accepted by all of the group's members; this scenario arises because the first collective result is produced by a mathematical analysis and without communication between the members of the group.

The remainder of this paper is organized as follows. Section 2 presents distance measures between the rankings. A new index for measuring the agreement level between two orders is described in Section 3. Section 4 presents an order-based consensus model for collaborative groups. Finally, in Section 5, we present our conclusions.

2. Distance measures between rankings

2.1. Related work

The problem of aggregating individual rankings to create an overall consensus ranking that is representative of the group has received much attention in the group decision-making literature. This problem arises in situations in which a group of decision makers (DMs) is asked to rank order a set of alternatives. The question is how to combine the DM rankings into one consensus ranking.

The problem of deriving a *consensus ranking* from preferences provided in pairwise formats was first examined by Kemeny and Snell (1962). They studied the group ranking problem with only preference rankings. The goal of their model is to minimize the number of reviewer ranking reversals. The problem of consensus ranking in a case in which preferences are represented in vector (rank order) format has been investigated extensively by many researchers, including Cook and Seiford (1978) and Cook and Kress (1991). Tavana et al. (2008) proposed a weighted-sum ordinal consensus ranking method in which the weights were derived from a sigmoid function. Hochbaum and Levin (2006) presented a model and algorithm for group rankings with intensity rankings, which generalizes the model of Ali et al. (1986), who presented an integer linear programming approach for consensus ranking. Recently, Cook et al. (2010) introduced a branch-and-cut algorithm to aggregate published journal rankings based on subsets of the accounting literature to create a consensus ranking.

In approaches that are related to multi-criteria decision analysis, we found that, in the late 1970s, Saaty (1977; 1980) developed the Analytic Hierarchy Process (AHP), which became an important approach to multi-criteria decision making. This technique has also been used in applications that require group rankings that use intensity rankings. Arrow and Raynaud (1986) considered the problem in which rankings that were provided, for example, by a group of evaluators must be combined into a common group ranking. In such a context, the authors suggested that a compromise ranking should be a prudent order. In general, a prudent order is not unique. Thus, Lamboray (2010) proposed a progressive refinement of the decision model and supports the group, eventually selecting one group ranking (a prudent order).

The group ranking problem has also been approached from the point of view of various outranking methods, such as those of Macharis et al. (1998), who proposed an extension of the PROMETHEE method for aiding group decisions. Leyva and Fernandez (2003) presented an extension of the ELECTRE III multi-criteria outranking methodology for assisting a group of decision makers.

On the other hand, Spearman's foot rule and Kendall's distance techniques fail to account for concepts that are crucial to evaluating a ranking in aiding group decision making, for example, the positional information. In other words, an inversion of an alternative in a high position should be more significant than an inversion of an alternative in a low position; inversions at the top of the ranking are costlier than inversions at the tail of the ranking. Yilmaz et al. (2008) and Carterette (2009) argued that the lack of this feature makes the Kendall distance a poor metric because it equally penalizes the inversions that are near the head and near the tail of a list. Recently, Kumar and Vassilvitskii (2010) presented a generalized version of the Kendall metric that captures element weights, position weights, and pairwise distances between permutations while, at the same time, retaining the classic form.

In accordance with these authors, we believe that, in aiding group decisions, an index that measures the agreement level between an individual opinion and a collective opinion, when both are expressed by rankings of a set of alternatives, can be more significant if it accounts for the following assumption: when we measure the divergence between two rankings, an inversion in a ranking of the alternatives that occupy the best ranks should be more penalized that an inversion of the alternatives that occupy the worst ranks. In the same context, Wang and Shen (1989) and Jabeur and Martel (2010) modified the Spearman correlation index (1904) and a distance measure developed by Roy and Slowinski (1993), respectively, to propose an agreement index between two vectors of ranks that account for the previous assumption.

2.2. Kendall's distance

Kendall's distance is a metric that counts the number of pairwise disagreements between two orders. The larger the distance, the more dissimilar the two orders are. Kendall's distance between two orders O^1 and O^2 is the following:

$$K(O^1,O^2)=\left|(i,j):i<j,(O^1(i)<O^1(j)\wedge O^2(i)>O^2(j))\vee(O^1(i)>O^1(j)\wedge O^2(i)<O^2(j))\right|$$

Kendall's distance can also be defined as $K(O^1,O^2)=\sum_{\{i,j\}\in P}\overline{K}_{i,j}(O^1,O^2)$ or, in an equivalent form, as the following:

$$K(O^1,O^2)=\sum_{i=1}^{m-1}\sum_{j>i}\overline{K}_{i,j}(O^1,O^2) \quad (1)$$

where

P is the set of unordered pairs of distinct elements in O^1 and O^2

$\overline{K}_{i,j}(O^1,O^2)=0$ if i and j are in the same order in O^1 and O^2

$\overline{K}_{i,j}(O^1,O^2)=1$ if i and j are in the opposite order in O^1 and O^2

This definition measures the total number of pairwise *inversions* between two orders O^1 and O^2, with $O(i)$ denoting the rank of element i.

$K(O^1,O^2)$ is equal to 0 if the two orders are identical and to $m(m-1)/2$ (where m is the order size) if one order is the reverse of the other. Kendall's distance is often normalized by dividing by $m(m-1)/2$, with the result that a value of 1 indicates the maximum disagreement. The normalized Kendall's distance therefore lies in the interval [0, 1].

$K(O^1, O^2)$ is a distance measure that quantifies the divergence between two orders. Based on the $K(O^1, O^2)$ formulation, we develop a first agreement index, called $P_A^{k,G}$, which measures the agreement level of an individual order O^k with respect to a collective temporary order O^G. Let:

$$P_A^{k,G} = 1 - \frac{2}{m(m-1)} K(O^k, O^G) = 1 - \frac{2}{m(m-1)} \sum_{i=1}^{m-1} \sum_{j>i} \overline{\mathrm{K}}_{i,j}(O^k, O^G) \quad (2)$$

where $O^k(r)\,(O^G(r))$ denotes the rank of element *r* in the individual order (in the collective temporary order).

Hence, the index $P_A^{k,G}$ is between 0 and 1. When the index $P_A^{k,G}$ is close to 1, then we say that the individual order O^k is similar to the collective temporary order O^G. In the opposite case, i.e., $P_A^{k,G}$ is close to 0, we say that these two orders are different.

Note that the index $P_A^{k,G}$ is symmetric, i.e., $P_A^{k,G} = P_A^{G,k}$, because it is based on Kendall's distance, which is a symmetric distance measure between orders; however, in the current form, the index $P_A^{k,G}$ is not necessarily the most appropriate measure for quantifying the agreement level of an individual order O^k with respect to a collective temporary order O^G. We illustrate this disadvantage with the following simple example: let R^G, R^1, R^2 be the collective temporary ranking and the individual rankings of decision makers 1 and 2, and let O^G, O^1, O^2 be the respective collective temporary order and the individual orders of decision makers 1 and 2, as shown in Figure 1.

$R^G = (a_4, a_2, a_5, a_1, a_3)$	$R^1 = (a_4, a_2, a_5, a_3, a_1)$	$R^2 = (a_2, a_4, a_5, a_1, a_3)$
$O^G = [4, 2, 5, 1, 3]$	$O^1 = [5, 2, 4, 1, 3]$	$O^2 = [4, 1, 5, 2, 3]$

Figure 1. Collective temporary ranking and individual rankings and their respective orders

By observing the individual orders of decision makers DM1 and DM2, called, respectively, O^1 and O^2, we expect that, when these are compared to the collective temporary order, DM2 is less satisfied than DM1, and consequently, the agreement level of the second will be lower than that of the first. This reasoning can be justified by the fact that DM2 does not have its "most preferred" alternative at the head of the collective temporary order, whereas "the best" alternative of DM1 is found at the top of the collective temporary order. However, by using the formula defined in (2) to compute the agreement index at each of the two DMs, they have the same index value: $P_A^{1,G} = P_A^{2,G} = 0.9$. Thus, we believe that an agreement index can be more significant if it accounts for the following assumption: an inversion in an order of the alternatives that occupies the best ranks should be more penalized when we measure the divergence between two orders than an inversion of the alternatives that occupies the worst ranks. In the same context, Wang and Shen (1989) and Jabeur and Martel (2010) modified the Spearman correlation index (1904) and a distance measure developed by Roy and Slowinski (1993), respectively, to propose an agreement index between two orders that accounts for the previous assumption.

3. An index for measuring the agreement level between two orders

3.1. A position-weighted version of Kendall's distance

With the aim of defining a measure that penalizes the inversions that are earlier in the order than the inversions that are late in the order, we therefore introduce the *position weights* to differentiate between inversions occurring near the head or the tail of an order.

To study the effect of the position weights, we first model the total cost of a swap between two different positions. Before presenting the index, we must first introduce the concept of the relative importance of an alternative in a given order.

Definition 1. The score of an alternative *i* in an order O^k, called s_i^k, is obtained by counting the number of alternatives that alternative *i* is preferred to plus one. The score s_i^k is known as the Borda score.

Definition 2. The relative importance of an alternative *i* in an order O^k, called δ_i^k, is the ratio of the score of alternative *i* on the sum of the scores of all of the alternatives of the order O^k. Let

$$\delta_i^k = \frac{s_i^k}{\sum_{i=1}^{m} s_i^k}$$

Note that $\delta_i^k \geq 0$. Define, for each alternative *i*, $1 \leq i \leq m$, $p_{O^k(i)} = \sum_{j=1}^{O^k(i)} \delta_j^k$. Let $\overline{p}_i(O^k) = \dfrac{p_{O^G(i)} - p_{O^k(i)}}{O^G(i) - O^k(i)}$ be the position weight *i*, which represents the average cost that alternative *i* encounters in moving from position $O^G(i)$ to position $O^k(i)$, where $\overline{p}_i = 1$ if $O^G(i) = O^k(i)$. By the monotonicity of the p_i's, we have $\overline{p}_i(O^k) > 0$ for all *i*.

Next, we proceed to define the position-weighted version of the Kendall distance ($K_{\overline{p}}$), as follows:

$$K_{\overline{p}}(O^k, O^G) = \sum_{i=1}^{m-1} \sum_{j>i} max\{\overline{p}_i(O^k), \overline{p}_j(O^k)\} \overline{K}_{i,j}(O^k, O^G) \quad (3)$$

where $\overline{p}_i(O^k)$ (or $\overline{p}_j(O^k)$) is the position weight of the alternative *i* (or the alternative *j*) in the individual order O^k. Note that, if $\forall i, \delta_i = 1$, (the relative importance of the alternatives is the same), then $\forall i, \overline{p}_i(O^k) = 1$. Hence, for unit swap costs, (3) collapses to (1).

The measure $K_{\overline{p}}$ is not symmetric because of the presence of the terms $\overline{p}_i(O^k)$ and $\overline{p}_j(O^k)$, which depend on the relative importance of the alternatives *i* and *j* - δ_i^k, δ_j^k - in the individual order O^k. Therefore, $K_{\overline{p}}$ is not a metric, in general. However, $K_{\overline{p}}$ is a versatile and convenient measure of the divergence between orders. By choosing the $\overline{p}_i(O^k)$ parameters to be the average cost that alternative *i* encounters in moving from one position to another position, we can emphasize the greater importance of ordering the first alternatives in O^k correctly relative to the correct order of the alternatives with low ranks in O^k.

3.2. Basic properties

The proposed evaluation measure satisfies some basic properties that make it easier to reason about.

i) The position-weighted Kendall's measure $K_{\overline{p}}$ is scale invariant.

Lemma 1. Given a position weight $\overline{p}$, let $\overline{p}' = c \cdot \overline{p}$ for a constant $c > 0$. Then, for any orders O^k, O^l, $K_{\overline{p}'}(O^k, O^l) = cK_{\overline{p}}(O^k, O^l)$.

Proof.

$$K_{\overline{p}'}(O^k, O^l) = \sum_{i=1}^{m-1} \sum_{j>i} \max\{\overline{p}_i'(O^k), \overline{p}_j'(O^k)\}\overline{K}_{i,j}(O^k, O^l)$$

$$= \sum_{i=1}^{m-1} \sum_{j>i} max\{c \cdot \overline{p}_i(O^k), c \cdot \overline{p}_j(O^k)\}\overline{K}_{i,j}(O^k, O^l)$$

$$= c\sum_{i=1}^{m-1} \sum_{j>i} max\{\overline{p}_i(O^k), \overline{p}_j(O^k)\}\overline{K}_{i,j}(O^k, O^l) = cK_{\overline{p}}(O^k, O^l)$$

ii) The position-weighted Kendall's measure $K_{\overline{p}}$ does not depend on the actual identity of the elements.

To compare two ordered sets (on the same set of alternatives), the approach of the position-weighted Kendall's measure is to sum the maximum between the position weight of the alternatives *i* and *j* of different pairs between these two ordered sets. There is a number to which we invariably arrive regardless of how we sum the maximum between the position weights of two alternatives of different pairs between two ordered sets. The quantity – a position-weighted Kendall's measure – is associated with two ordered sets (on the same set of alternatives) and is invariant under the process of summing the maximum between the position weights of two alternatives of different pairs between two ordered sets.

iii) $K_{\overline{p}}$ satisfies three of the four axioms for a metric.

We assume $\overline{p}_i > 0$, $K_{\overline{p}}(O^k, O^G) \geq 0$ and $K_{\overline{p}}\left(O^k, O^G\right) = 0$ if and only if $O^k = O^G$. For the triangle inequality, we first show a reduction from the position-weighted case to the unweighted case.

Lemma 2. Given $\overline{p}$, for any orders σ_1, σ_2 there are orders τ_1, τ_2 such that $K_{\overline{p}}(\sigma_1, \sigma_2) = K(\tau_1, \tau_2)$.

Proof. Without loss of generality, we can suppose that the $\overline{p}_i$ are assumed to be integers. Consider the set of inversions $\langle i, j \rangle$ produced between σ_1, σ_2. Each inversion $\langle i, j \rangle$ has an associated number $k^{\langle i,j \rangle} = \max\{\overline{p}_i, \overline{p}_j\}$. Suppose that there are *n* inversions, which are ordered as they appear, according to the application of the measure $K_{\overline{p}}$; then, we have a set $\{k_1^{\langle i,j \rangle}, k_2^{\langle i,j \rangle}, \ldots, k_n^{\langle i,j \rangle}\}$ of integer numbers in which possibly some $k_r^{\langle i,j \rangle}$ are repeated.

We can associate an inversion $\langle i, j \rangle$ of the position weight $k_r^{\langle i,j \rangle} > 0$ with the $k_r^{\langle i,j \rangle}$ alternatives a_i that are placed after the first $k_1^{\langle i,j \rangle}, k_2^{\langle i,j \rangle}, \ldots, k_{r-1}^{\langle i,j \rangle}$ alternatives a_i of an arbitrary order τ_1 of the canonical order$[a_1, a_2, \ldots, a_M]$, where M is a sufficiently large number. Here, τ_1 retains all of the alternatives associated with an inversion in the same order and reorders the blocks based on the order of appearance of an inversion in $K_{\overline{p}}$.

Consider an order τ_2 for which the first $k_1^{\langle i,j \rangle} + k_2^{\langle i,j \rangle} + \ldots + k_n^{\langle i,j \rangle} - 1$ alternatives are the first $k_1^{\langle i,j \rangle} + k_2^{\langle i,j \rangle} + \ldots + k_n^{\langle i,j \rangle} - 1$ alternatives of τ_1, beginning from the second position, and the position $k_1^{\langle i,j \rangle} + k_2^{\langle i,j \rangle} + \ldots + k_n^{\langle i,j \rangle}$ contains the alternative that is in position one of τ_1. The remaining alternatives of τ_1 are placed in the same position in τ_2 .Then, $K_{\overline{p}}(\sigma_1, \sigma_2) = K(\tau_1, \tau_2)$. ∎

It is important to note that the position weights depend on the orders σ_1, σ_2 that are being considered.

3.3. The new agreement index

Note that the above lemma shows that $K_{\bar{p}}$ satisfies the triangle inequality. We are now ready to define the position-weighted version of the index $P_A^{k,G}$:

Definition 3. Let O^G be a collective temporary order, and let O^k be an individual order. The index that accounts for the position weights of the alternatives in O^k to quantify the agreement level of O^k with respect to O^G is defined as follows:

$$wP_A^{k,G} = 1 - \frac{m(m+1)}{\left\lfloor \frac{m}{2} \right\rfloor (2m^2 + 3m - (2m\left\lfloor \frac{m}{2} \right\rfloor + 4\left\lfloor \frac{m}{2} \right\rfloor + 2))} \sum_{i=1}^{m-1} \sum_{j>i} \max\{\bar{p}_i(O^k), \bar{p}_j(O^k)\} \bar{K}_{i,j}(O^k, O^G)$$

(4)

where $\bar{p}_i(O^k)$ (or $\bar{p}_j(O^k)$) is the position weight of the alternative *i* (or the alternative *j*) in the individual order O^k and $\lfloor x \rfloor$ is the integer part of a positive real number *x*. Note that the index $wP_A^{k,G}$ is not symmetric because of the presence of the terms $\bar{p}_i(O^k)$ and $\bar{p}_j(O^k)$, which depend on the relative importance of the alternatives *i* and *j* - δ_i^k, δ_j^k - in the individual order O^k.

Proposition 1. Let O^G be a collective temporary order, and let O^k be an individual order. Then, it is possible to verify the following:

1.1. If both orders O^G and O^k are identical, then $wP_A^{k,G} = 1$;
1.2. If both orders O^G and O^k are completely opposite, then $wP_A^{k,G} = 0$;
1.3. $0 \le wP_A^{k,G} \le 1$,
1.4. The index $wP_A^{k,G}$ verifies the monotonicity property, i.e., this index increases (decreases) when the individual order O^k approaches (moves away from) the collective temporary order O^G.

Proof.

1.1. When the orders O^k and O^G are identical, then $\forall i, j \ (i = 1,2,\ldots, m-1,\ j > i)$, we have $\bar{K}_{i,j}(O^k, O^G) = 0$ because (*i,j*) are in the same order in O^k and O^G. Thus, $wP_A^{k,G} = 1$.

1.2. We should emphasize that the largest possible distance between two orders is obtained when one order is the exact reverse of the other order. Without loss of generality, we suppose that the individual and the collective temporary orders are as follows:

Collective temporary order: $a_1 \succ a_2 \succ a_3 \succ \ldots \succ a_m$

Individual order: $a_m \succ a_{m-1} \succ a_{m-2} \succ \ldots \succ a_1$

Based on the above two orders, the part of the summation of the agreement index presented in formula (4) can be written in the following way:

$$\sum_{i=1}^{m-1} \sum_{j>i} \max\{\bar{p}_i(O^k), \bar{p}_j(O^k)\} \bar{K}_{i,j}(O^k, O^G) = \sum_{i=1}^{\left\lfloor \frac{m}{2} \right\rfloor} (m+1-2i)\bar{p}_i + \sum_{i=\left\lfloor \frac{m}{2} \right\rfloor + 2}^{m} (2i-m-2)\bar{p}_i$$

$$= \sum_{i=1}^{\left\lfloor \frac{m}{2} \right\rfloor} (m+1-2i) \sum_{j=i+1}^{m-i+1} \frac{\delta_j}{m+1-2i} + \sum_{i=\left\lfloor \frac{m}{2} \right\rfloor+2}^{m} (2i-m-2) \sum_{j=m+2-i}^{i} \frac{\delta_j}{2i-m-1}$$

$$= \sum_{i=1}^{\left\lfloor \frac{m}{2} \right\rfloor} \frac{m+1-2i}{m+1-2i} \sum_{j=i+1}^{m-i+1} \delta_j + \sum_{i=\left\lfloor \frac{m}{2} \right\rfloor+2}^{m} \frac{2i-m-2}{2i-m-1} \sum_{j=m+2-i}^{i} \delta_j$$

$$= \sum_{i=1}^{\left\lfloor \frac{m}{2} \right\rfloor} \sum_{j=i+1}^{m-i+1} \delta_j + \sum_{i=\left\lfloor \frac{m}{2} \right\rfloor+2}^{m} \frac{2i-m-2}{2i-m-1} \sum_{j=m+2-i}^{i} \delta_j$$

$$= \sum_{i=1}^{\left\lfloor \frac{m}{2} \right\rfloor} \sum_{j=i+1}^{m-i+1} \frac{2j}{m(m+1)} + \sum_{i=\left\lfloor \frac{m}{2} \right\rfloor+2}^{m} \frac{2i-m-2}{2i-m-1} \sum_{j=m+2-i}^{i} \frac{2j}{m(m+1)}$$

$$= \frac{2}{m(m+1)} \sum_{i=1}^{\left\lfloor \frac{m}{2} \right\rfloor} \sum_{j=i+1}^{m-i+1} j + \sum_{i=\left\lfloor \frac{m}{2} \right\rfloor+2}^{m} \left(\frac{2i-m-2}{2i-m-1}\right)\left(\frac{2}{m(m+1)}\right) \sum_{j=m+2-i}^{i} j$$

Using the general formula for the sum of an arithmetic progression $S_n = a_1 + a_2 + ... + a_n = \frac{n(a_1 + a_n)}{2}$, we have

$$= \frac{m^2\left\lfloor \frac{m}{2} \right\rfloor + 2m\left\lfloor \frac{m}{2} \right\rfloor - m\left\lfloor \frac{m}{2} \right\rfloor\left\lfloor \frac{m}{2} \right\rfloor - 2\left\lfloor \frac{m}{2} \right\rfloor\left\lfloor \frac{m}{2} \right\rfloor}{m(m+1)} + \frac{m^2\left\lfloor \frac{m}{2} \right\rfloor - m\left\lfloor \frac{m}{2} \right\rfloor\left\lfloor \frac{m}{2} \right\rfloor + m\left\lfloor \frac{m}{2} \right\rfloor - 2\left\lfloor \frac{m}{2} \right\rfloor\left\lfloor \frac{m}{2} \right\rfloor - 2\left\lfloor \frac{m}{2} \right\rfloor}{m(m+1)}$$

$$= \frac{\left\lfloor \frac{m}{2} \right\rfloor (2m^2 + 3m - (2m\left\lfloor \frac{m}{2} \right\rfloor + 4\left\lfloor \frac{m}{2} \right\rfloor + 2))}{m(m+1)}$$

In this case, the agreement index value is:

$$wP_A^{k,G} = 1 - \frac{m(m+1)}{\left\lfloor \frac{m}{2} \right\rfloor (2m^2 + 3m - (2m\left\lfloor \frac{m}{2} \right\rfloor + 4\left\lfloor \frac{m}{2} \right\rfloor + 2))} \left(\frac{\left\lfloor \frac{m}{2} \right\rfloor (2m^2 + 3m - (2m\left\lfloor \frac{m}{2} \right\rfloor + 4\left\lfloor \frac{m}{2} \right\rfloor + 2))}{m(m+1)} \right)$$

$$wP_A^{k,G} = 0$$

1.3. This statement is trivial in view of the fact that it is a corollary of (1.1) and (1.2).

1.4. The index $wP_A^{k,G}$ verifies the monotonicity property, i.e., this index increases (decreases) when the individual order O^k approaches (moves away from) the collective temporary order O^G.

Thus, to demonstrate the monotonicity of the index $wP_A^{k,G}$, we should prove that, if an individual order approaches (or moves away from) a collective temporary order in terms of the distance and position weight of the alternatives, then its agreement index increases (or decreases). For this purpose, let us consider two individual orders O^k and O^{k^*} and a collective temporary order O^G, in which we assume that the order O^{k^*} is obtained by modifying the order O^k for the purpose of making O^{k^*} closer to O^G than O^k. Without loss of generality, we prove that, if order O^{k^*} is closer (in terms of the distance and position weight of the alternatives) than the order O^k to the order O^G, then we should have $wP_A^{k^*,G} \geq wP_A^{k,G}$.

Thus, when O^{k^*} is closer than O^k to O^G, we have $\forall i, j, i = 1,2,\ldots,m-1, \ j > i$: $\overline{K}_{i,j}(O^k, O^G) \geq \overline{K}_{i,j}(O^{k^*}, O^G)$ and $Max\{\overline{p}_i(O^k), \overline{p}_j(O^k)\} \geq Max\{\overline{p}_i(O^{k^*}), \overline{p}_j(O^{k^*})\}$ because $\left|p_{O^G(i)} - p_{O^k(i)}\right| \geq \left|p_{O^G(i)} - p_{O^{k^*}(i)}\right|$ and $\left|p_{O^G(j)} - p_{O^k(j)}\right| \geq \left|p_{O^G(j)} - p_{O^{k^*}(j)}\right|$.

On the basis of the above inequalities and the agreement index formula defined in (4), we can easily deduce that $wP_A^{k^*,G} \geq wP_A^{k,G}$. $\blacksquare$

When the formula of the proposed agreement index $wP_A^{k,G}$ is applied to the numerical example presented in Figure 1, we obtain the following results: $wP_A^{1,G} = 0.943$, $wP_A^{2,G} = 0.857$.

Although there is only one permutation in each of the two individual orders with respect to the collective temporary order, we observe that decision maker 2 is less satisfied than decision maker 1 because $wP_A^{2,G} \leq wP_A^{1,G}$. This result could be justified by the fact that decision maker 2 does not find his "best" alternative (i.e., a_2) in the head of the collective temporary order, whereas the "best" alternative of decision maker 1 (i.e., a_4) is the same alternative as that of the collective temporary order.

4. A Consensus model

After the group ranking is created, we evaluate how good it is by checking whether it represents the majority of the group members' preferences. It is very rare for all of the individuals in a group to share the same opinion about the alternatives because a diversity of preferences commonly exists.

4.1. A new consensus measure model

In this section, we present a new consensus measure model. Initially, in any non-trivial group multi-criteria ranking problem, the decision makers disagree in their preferences so that the consensus must be viewed as an iterative process, which means that agreement is obtained only after some rounds of consultation. In each round, we calculate two consensus parameters, a *consensus measure* and a *proximity measure*. The first parameter guides the consensus process, and the second parameter supports the group discussion phase of the consensus process. The main problem is how to find a way of making the individual positions converge and, therefore, how to support the decision makers in obtaining and agreeing with a specific solution. To accomplish this goal, a consensus level α required for that solution is fixed in advance ($\alpha \in [0,1], \alpha > 0.5$). When the consensus measure reaches this level, then the decision-making session is finished and the solution is obtained. If that scenario does not occur, then the decision-makers' preferences must be modified. This modification is accomplished in a group discussion session in which we use a proximity measure to propose a feedback process based on simple rules, which supports the decision makers in changing their preferences.

The consensus model for this group multi-criteria ranking problem will be described in further detail in the following subsections.

4.1.1. Consensus and proximity measures

The proposed model considers the weighted measure $wP_A^{k,G}$ of the previous section for expressing the differences in the ranking discrepancies between the group temporary ranking and the individual rankings. Usually, we consider the rankings of a set of alternatives in some order, typically from the most important to the least important. To use the index $wP_A^{k,G}$, it is first necessary to convert the difference of the ranking into numerical data. Thus, we use the equivalent representation of a ranking as a list of ranks $O^i = [o^i(1),...,o^i(m)]$, showing the position of alternative *j* in the ranking (Chiclana et al. 1998; Seo and Sakawa 1985). Therefore, according to this point of view, an ordered vector of alternatives from best to worst is given.

Suppose that the *i-th* individual has an associated ranking of R_i. If one wishes to evaluate how different the derived ranking R_i is with regard to the group temporary ranking R_G, then the derived ranking could be measured with a proximity measure.

Each consensus parameter requires the use of the index $wP_A^{k,G}$ to obtain the level of agreement between the individual solution of decision maker *i*, $O^i = [o^i(1),...,o^i(m)]$, and the collective solution $O^G = [o^G(1),...,o^G(m)]$.

We define consensus indicators by comparing the positions' weighted rank of alternatives in two preferences vectors, as follows:

1. We use a multi-criteria decision making method (e.g., Electre III, Promethee II) to obtain the individual rankings R_i for each decision maker; then, we use a group multi-criteria decision making method (Electre GD, Promethee for groups) to obtain a collective ranking of alternatives R_G.

2. We calculate the ordered vector of alternatives $\{O^i; i = 1,2,...,n\}$ (from the individual rankings R_i) and the collective ordered solution O^G (from the collective ranking R_G), where *n* is the number of decision makers in the group.

3.

3.1 First, we calculate the position-weighted version of the Kendall distance $K_{\bar{p}}$ defined in (3). The agreement function used allows for a flexible matching between the individual solution and the collective temporary solution.

3.2 We calculate the proximity of the i-th decision maker's individual solution to the collective temporary solution, called $wP_A^{k,G}$, by using the expression (4).

When the proximity value associated with the *i-th* decision maker is close to 1, his contribution to the consensus is high (positive), while if it is close to 0, then that decision maker has a negative contribution to the consensus.

4. The *global consensus measure*, called C_A, is calculated by the aggregation of the above consensus degrees for each decision maker.

We calculate the consensus degree of all of the decision makers using the following expression:

$$C_A = \sum_{i=1}^{n} \frac{wP_A^{i,G}}{n}$$

where *n* is the number of decision makers in the group.

4.1.2. Interactive procedure for reaching a consensus order

A collective order, obtained from a first aggregation of the individual orders, may be not accepted by some members of the group. Typically this scenario could result from the fact that this first collective result is determined by a mathematical analysis and without communication between the members of the group. In other words, the consensus measure C_A

has not reached the consensus level required; then, the decision-makers' rankings must be modified. In these cases, it is essential that the members have an interactive communication and cooperation that enables them to reach, from one collective temporary order and their individual order, a consensus order. As we stated earlier, we are using the proximity measures $wP_A^{k,G}$ to build a feedback process so that decision makers can adjust their preferences to achieve closer preferences between them. This feedback mechanism will be applied when the consensus level is not satisfactory and will be finished when a satisfactory consensus level is reached.

Essentially, the proposed procedure has two main phases. The first phase is a checking phase in which the members of the group determine, on the basis of their agreement indices, whether or not the current collective order can be qualified as a consensus. If this test is negative, then the members will be engaged in an interactive phase (second phase) to exchange information and revise their individual rankings. These revisions will allow them to reduce their divergences with the current collective order, and consequently, a consensus order might be reached.

The interactive procedure that supports the members in reaching a consensus order O^C is presented as follows:

1. Compute, for each member *k*, the agreement level $wP_A^{k,G}$ of this order O^k with respect to the collective temporary order O^G
2. Compute the global consensus measure C_A
3. If the global consensus measure C_A exceeds the predefined threshold α expressing the majority principle (often equal to 2/3 or 3/4), then the collective temporary order O^G is qualified as a consensus, and the procedure is stopped because a consensus order has been reached. In the opposite case, i.e., the collective temporary order has a global consensus measure that does not exceed the threshold α, then we move to step 4 (the feedback process);
4. In this step, we propose to the members of the group a feedback process in which they can exchange information and discuss and modify their ranking to reach a consensus order. For this purpose, we proceed as follows:
   4.1. Identify the members *k* whose proximity measure $wP_A^{k,G}$ is less than a predetermined threshold ρ; then, these members must change their preferences.
   4.2. Propose that these members make concessions that allow them to reduce their divergences with the collective temporary order. This revision will increase the chance of each of them of agreeing with the collective temporary order in a subsequent iteration. Indeed, we believe that it is essential to present the collective temporary order to the members so that each of them identifies his sources of divergence with this order. When the individual orders are modified, the aggregation procedure is again performed to produce a new collective temporary order O^G. Return to step 1.

It is important to emphasize that the success or the failure of this interactive procedure depends mainly on the members' degree of commitment, as expressed by their concession efforts, to accomplish the decision-making process. Additionally, the consensus-reaching process will depend on the size of the group of decision makers as well as on the size of the set of alternatives; when these sizes are small and when preferences are homogeneous, the consensus level required is easier to obtain. However, we note that the change in preferences can produce a change in the temporary collective solution, especially when the decision makers' preferences are quite different, i.e., in the early stages of the consensus process. In fact, when decision makers' preferences are close, i.e., when the consensus measure approaches the consensus level that is required, the changes in the decision makers' preferences will not affect the temporary collective solution; it will only affect the consensus measure. This process is convergent to the collective solution once the consensus measure is sufficiently high.

5. Conclusions

In this paper, we proposed an index for measuring the agreement level of an individual order with respect to a collective order. This index constitutes an interesting weighted version of the well-known Kendall's rank correlation index. It accounts for the position weights of the alternatives in an individual order to quantify the agreement level of an individual order with respect to a collective temporary order. We consider that this characteristic provides a better evaluation of the agreement level of each individual order with respect to the collective temporary order because an inversion in a ranking of the alternatives occupying the best rank are more penalized that an inversion of the alternatives occupying the worst ranks, i.e., it focuses on the "best" alternatives.

To show one application of this agreement index during a decision-making process, this paper includes an order-based consensus model for collaborative groups. This model is based on the outranking approach for modeling individual and group preferences. Based on the proposed agreement index, we define consensus and proximity measures; the consensus measure guides the consensus process, and the proximity measure supports the group discussion phase of the consensus process. This model includes an interactive procedure for reaching a consensus order. In particular, the model generates advice on how decision makers should change their preferences to reach a ranking of alternatives with a high degree of consensus.

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