A COMPENSATORY INFERENCE SYSTEM

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ABSTRACT

This paper aims to introduce a new and useful inference system based on CFL. It will be called Compensatory Inference System (CIS). It is a generalization of the deduction like in mathematical logic, with one implication operator found in the literature. It is an octet (L,Ax,R,*c*,*d*,*o*,*n*,→), where *c*, *d*, *o* and *n* are the operators of the Compensatory Logic on Geometric Mean, L is the set of formulas of the propositional calculus in this CFL, Ax is the Kleene's logical axioms, R is the modus ponens rule and → is the Reichenbach implication. This combination maximizes the validity of the Kleene's axioms, as it will be exposed through this chapter. Therefore, it is an inference system with easy application and logically rigorous.

KEYWORDS: Fuzzy Inference Systems, Compensatory Fuzzy Logic, Mathematical Fuzzy Logic

Introduction

There are many approaches to the inference in fuzzy logic. They can be found in both *Fuzzy Logic in the Narrow Sense* and *Fuzzy Logic in the Broad Sense*.

The *Fuzzy logic in the narrow/technical sense* refers to syntax, semantic, axiomatization, completeness and other formalizations, proper of every many-valued logic. *Fuzzy logic in the broad* or *wide sense* is a singularity of this many-valued logic and refers to concepts like linguistic variable, fuzzy if-then rule, fuzzy quantification and defuzzification, truth qualification, the extension principle, the compositional rule of inference and interpolative reasoning, etc. (Dubois et al., 2007).

The *fuzzy if-then rules* are a much appreciated tool of fuzzy logic; it is the basis of the *fuzzy reasoning* and the *fuzzy inference systems*. *Fuzzy if-then rules* are conditional statements of the form "if x_1 is A_1 and …and x_n is A_n, then y is B", where A_i and B are fuzzy sets. The inference is realized by using the *Generalized Modus Ponens*, which is basically the classical modus ponens, but particularly consists of: from "x is A*" and "if x is A then y is B" we infer B*. The fuzzy set B* has membership function with formula that can be found more detailed in (Jang et al., 1997). For example, from the sentence "the tomato is more or less red" and the rule "if the tomato is red, then it is ripe", it can be inferred that "the tomato is more and less ripe", according to the generalized modus ponens. The Generalized Modus Ponens is also known as *Fuzzy Reasoning* or *Approximate Reasoning*.

The *Fuzzy Inference System*s are successful frameworks for inference calculation. The *Mamdani Fuzzy Models*, the *Sugeno Fuzzy Models* and the *Tsukamoto Fuzzy Models* are three of them, very recurrent in the literature.

The basic scheme of those *Fuzzy Inference System*s is: from a set of crisp or fuzzy input data, other fuzzy sets are obtained as a consequence of every fuzzy if-then rule. The next step is the use of an aggregator, which is a unique function, representing the results of the conjoint of rules. This function needs to be defuzzified, that is to say, the fuzzy results are converted to a single crisp output value.

The success in the application of the above tools is guaranteed, but they include some extra-logical methods, including the defuzzification. Besides, the functions defined as aggregators in the models aren't well justified from the point of view of the classical logics.

The inference upon the Fuzzy Logic in Narrow Sense has some interesting results that are more related to the mathematical logic. The concept of *lattice*, which is defined in algebra and set theory, is the point of departure. A *lattice* is a particular case of *partially ordered set*; therefore, it includes a binary relation of order over every pair of elements of the set. The logic systems found in the literature are based on t-norm and t-conorm.

The axiomatic of the *Propositional fuzzy logic,* also known as *Basic fuzzy logic* or *Basic fuzzy propositional logic*, was introduced by Hájek in 1998. It is a Hilbert-style deduction system with the classical modus ponens as the unique rule of inference (Dubois et al., 2007). A natural extension of this axiomatic is obtained when including some axioms, where the universal and existential quantifiers appear; it is the *Basic fuzzy predicate logic*. In Hilbert-style deduction systems the axiomatic is formed by formulas with the implication operator. The Basic fuzzy logic takes the operators of implication and conjunction, and the constant 0 (false) for defining the negation operator and the disjunction operator.

The *Fuzzy logic with evaluated syntax* includes the notion of lattice. Every antecedent formula is directly evaluated with its truth-value, and the result of the deduction, obtained with the application of the modus ponens, is a formula with a truth-value too, computed by the order of the lattice. The axioms not fully true are equally accepted; see (Nóvak and Dvorák, 2008). The deduction in the Fuzzy logic with evaluated syntax uses the theory of proof of classical logic, hence, it is a natural generalization of the mathematical logic.

However, Vilém Nóvak in (Nóvak, 2012) agreed that these calculi in the narrow sense haven't enough variety of applications.

Other approaches and tendencies about the inference can be found in (Dubois et al., 2007; Nóvak and Dvorák, 2008; Nóvak, 2012).

The apparent dichotomy, which means the use of mathematically rigorous calculi not widely applied or the use of pragmatic tools for inference with many successful applications, is perhaps due to the use of the t-norm and t-conorm paradigm as the unique option to define fuzzy logic systems. However, the compensatory operators seem to be more adequate to model the human thinking, according to some experiments (Mizumoto, 1989). Also, some calculations like defuzzification, essentially compute a mean.

The unique paradigm of fuzzy logic system found in the literature, based on compensatory operators, and not exclusively on single isolated operators (Detyniecki, 2001), is the *Compensatory Fuzzy Logic* (CFL) (Espin et al., 2011). A CFL system is a quartet of: a conjunction operator, a disjunction operator, a negation operator and a strict fuzzy order operator. They must satisfy an axiomatic that will be detailed in sections below, belonging to the logic and the decision theory.

CFL is a development of a logic system, which serves as a natural bridge between Fuzzy logic in narrow sense and in broad sense. This is because this axiomatic may be considered an extension of the mathematical logic, and also, it can be successfully associated with disciplines of the Artificial Intelligence, according to the notion of *Soft Computing* (Zadeh, 1998), or to the methodology for *Experts Systems*, called *Knowledge Engineering* (Buchanan and Shortliffe, 1984).

The CFL makes possible the computation by words and not exclusively by numbers, as proposed by Zadeh (Zadeh, 2002); the particularity of the CFL is that it doesn't use only simple linguistic variables, but complex phrases expressed in natural language. Thus, a CFL allows to model problems expressed in natural language, using sentences provided by experts in the theme, following the methodology of the Expert Systems. The uncertainty is another characteristic of this logic system, typical of fuzzy logic.

This chapter aims to introduce a new and useful inference system based on CFL. It will be called Compensatory Inference System (CIS). It is a generalization of the deduction like in mathematical logic, with one implication operator found in the literature.

The chapter is organized as follows: the epigraph called Preliminaries explains the basic theory about CFL and introduces a one-parameter family system; in the second part, the theory of implication operators is explained and six of them are selected for the CIS, according to some criteria. Compensatory Inference Systems is the name of the epigraph that exposes the criteria and the results about the possible CIS, and finally, the association of an implication operator and a representative of the family form the definitive CIS, selected with the aid of an optimization computation.

Preliminaries

Compensatory Fuzzy Logic

A CFL system is a quartet (c,d,o,n) of operators of conjunction, disjunction, fuzzy strict order and negation, respectively (Espin et al., 2011).

c and d map vectors of $[0,1]^n$ into $[0,1]$, o is a mapping from $[0,1]^2$ into $[0,1]$, and n is a unary operator of $[0,1]$ into $[0,1]$.

The following axiomatic must to be satisfied:

i. Compensation Axiom: $\min(x_1,x_2,...,x_n) \le c(x_1,x_2,...,x_n) \le \max(x_1,x_2,...,x_n)$

ii. Commutativity or Symmetry Axiom: $c(x_1,x_2,...,x_i,...,x_j,...,x_n)= c(x_1,x_2,...,x_j,...,x_i,...,x_n)$

iii. Strict Growth Axiom: If $x_1=y_1$, $x_2=y_2$, ..., $x_{i-1}=y_{i-1}$, $x_{i+1}=y_{i+1}$,...., $x_n=y_n$ are unequal to zero, and $x_i>y_i$ then $c(x_1,x_2,...,x_n)>c(y_1,y_2,...,y_n)$

iv. Veto Axiom: If $x_i=0$ for one i, then $c(\mathbf{x}) =0$.

v. Fuzzy Reciprocity Axiom: $o(\mathbf{x},\mathbf{y}) = n[o(\mathbf{y},\mathbf{x})]$

vi. Fuzzy Transitivity Axiom: If $o(\mathbf{x},\mathbf{y}) \ge 0.5$ and $o(\mathbf{y},\mathbf{z}) \ge 0.5$, then

$o(\mathbf{x}, \mathbf{z}) \ge \max(o(\mathbf{x},\mathbf{y}),o(\mathbf{y},\mathbf{z}))$

vii. De Morgan's Laws:

$n(c(x_1,x_2,...,x_n)) = d(n(x_1),n(x_2),...,n(x_n))$ and $n(d(x_1,x_2,...,x_n)) = c(n(x_1),n(x_2),...,n(x_n))$

A consequence of the axiomatic below is the following set of properties for the disjunction operator:

1. Compensation Property:
$\min(x_1,x_2,...,x_n) \le d(x_1,x_2,...,x_n) \le \max(x_1,x_2,...,x_n)$
2. Symmetry Property: $d(x_1,x_2,...,x_i,...,x_j,...,x_n) = d(x_1,x_2,...,x_j,...,x_i,...,x_n)$
3. Strict Growth Property:

If $x_1 = y_1, x_2 = y_2,..., x_{i-1} = y_{i-1}, x_{i+1} = y_{i+1},..., x_n = y_n$ are not equal to one, and $x_i > y_i$, then $d(x_1,x_2,...,x_n) > d(y_1,y_2,...,y_n)$

4. If $x_i=1$ for an i then $d(\mathbf{x})=1$

For a vector (a,a,…,a) at any $a\in[0,1]$, from the compensation axiom follows that $a = \min(a,a,...,a) \le c(a,a,...,a) \le \max(a,a,...,a) = a$. This result allows the conclusion by means of one of De Morgan's laws that the disjunction satisfies the same inequality. Therefore, the following Idempotency Property is met:

5. $c(a,a,…,a)=a$, $d(a,a,…,a)=a$

A family of CFL systems may be obtained from the quasi-arithmetic means, with the following formula below (Mitrinovic, 1993):

$$M_f(x_1,x_2,...,x_n) = f^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} f(x_i)\right) \qquad (1)$$

Where f(x) is a continuous and strictly monotonic function of one real variable.

In this chapter the one-parameter family with formula:

$$M_f(x_1,x_2,...,x_n) = \left(\frac{1}{n}\sum_{i=1}^{n} x_i^p\right)^{\frac{1}{p}} \qquad (2)$$

Where $p\in(-\infty,0]$ satisfies the axiom of compensation, if the conjunction is defined as follows:

$$c(x_1,x_2,...,x_n) = \left(\frac{1}{n}\sum_{i=1}^{n} x_i^p\right)^{\frac{1}{p}} \qquad (3)$$

The disjunction is defined as the dual of the conjunction, that is to say:

$$d(x_1,x_2,\dots,x_n)=\left(\frac{1}{n}\sum_{i=1}^{n}(1-x_i)^p\right)^{\frac{1}{p}} \quad (4)$$

The fuzzy negation is:

n(x) = 1-x (5)

The fuzzy strict order is:

$$o(\mathbf{x},\mathbf{y})=0.5[c(\mathbf{x})-c(\mathbf{y})]+0.5 \quad (6)$$

Let us remark that this is a special case of the formula (1), where f(x) = x^p, always that p<0. If p = 0, f(x) = ln(x), which corresponds to the geometric mean, and for p = -1 (3) represents the harmonic mean. Even though the limit $p=-\infty$ is not included in the definition above, its limit value exists and it is $c(x_1,x_2,\dots,x_n)=\min(x_1,x_2,\dots,x_n)$. Also, $\lim_{x_i\to 0} M_f(x_1,x_2,\dots,x_n)$ = 0 for all p, therefore,

$M_f(x_1,x_2,\dots,x_n)$ may be redefined as 0, if some x_i=0.

An interesting result is that for a fixed vector $(x_1,x_2,\dots,x_n)$, if s<p then in (2) the inequality $\left(\frac{1}{n}\sum_{i=1}^{n}x_i^s\right)^{\frac{1}{s}} \le \left(\frac{1}{n}\sum_{i=1}^{n}x_i^p\right)^{\frac{1}{p}}$ is fulfilled (Mitrinovic, 1993). Therefore, the parameter p indicates the behavior toward preferences; the decisor is more 'pessimistic' if he/she measures its preferences using a smaller parameter p.

Other formulas of the CFL are as follows:

$$\underset{x\in U}{\forall} p(\mathbf{x}) = \underset{x\in U}{\wedge} p(\mathbf{x}) = M_f(x_1,x_2,\dots,x_n) = f^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} f(x_i)\right) \quad (7)$$

$$\underset{x\in U}{\exists} p(\mathbf{x}) = \underset{x\in U}{\vee} p(\mathbf{x}) = d(x_1,x_2,\dots,x_n) = 1-f^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} f(1-x_i)\right) \quad (8)$$

If U is a continuous set, formulas (7) and (8) become (9) and (10), respectively:

$$\forall \mathbf{x}\, p(\mathbf{x}) = f^{-1}\left(\frac{\int_X f(p(\mathbf{x}))d\mathbf{x}}{\int_X d\mathbf{x}}\right) \quad (9)$$

$$\exists \mathbf{x}\, p(\mathbf{x}) = 1-f^{-1}\left(\frac{\int_X f(1-p(\mathbf{x}))d\mathbf{x}}{\int_X d\mathbf{x}}\right) \quad (10)$$

Where p(**x**) is a formula of the propositional calculus in CFL.

This formula is valid in the CFL if it satisfies the condition (11), below:

$$f^{-1}\left(\frac{\int_{[0,1]^n} f(p(\mathbf{x}))d\mathbf{x}}{\int_{[0,1]^n} d\mathbf{x}}\right) > \frac{1}{2} \quad (11)$$

Implication Operators

In fuzzy literature the classification of implication operators is usually defined using other operators, like conjunction, disjunction and negation, but they are always based on t-norm and t-conorm paradigm. In this chapter, these concepts will be extended to any fuzzy system, including the compensatory ones. Here, when it would be necessary, the operators will preserve their exact definition, even if they don't correspond to any classification and taking into account that often the definition of an implication operator is associated with a specific t-norm and t-conorm.

The criteria for selecting implication operators for our purposes are the following:

1. The operator satisfies the truth-value table of the bivalent classical logic, when the truth-values calculus is restricted only to the set {0, 1}. Briefly, the truth-value of the formula $x \rightarrow y$ is 1 if $x = 0$ or $x = y = 1$, and is 0 if $x = 1$ and $y = 0$.

2. The operator must be a continuous function with regard to both arguments or have a finite number of removable discontinuities.

The reason for imposing condition 1 is that the CIS must be a natural extension of the mathematical logic. Whereas condition 2 guarantees the 'sensitiveness' of the CIS, that is to say, any change in the simple predicates will be reflected in the final results of their corresponding composed predicates.

Some classifications appearing in the literature are:

- S-implication (Dubois et al., 2007): $I_S(x,y) = d(n(x),y)$, where d and n are the disjunction and negation operators, respectively.
- R-implication (Dubois et al., 2007): $I_R(x,y) = \sup\{z \in [0,1]: c(x,z) \leq y\}$, where c is the conjunction operator.
- QM-implication (Trillas et al., 2000), which is also known as QL-implication (Dubois et al., 2007): $I_{QL}(x,y) = d(n(x),c(x,y))$

A-implication (Turksen et al., 1998): The operator satisfies a group of axioms, which implicitly associate it with the conjunction, disjunction and negation operators. For example, the Law of Importation $(x \wedge y \rightarrow z) \leftrightarrow (x \rightarrow (y \rightarrow z))$ is one of its axioms, where the symbol $\leftrightarrow$ is the logic equivalence.

The implication operators suggested in the literature satisfy the two conditions expressed above, and their classifications are:

- Reichenbach implication (S-implication): $x \rightarrow y = 1 - x + xy$
- Klir-Yuan implication (a variation of the above case without a classification): $x \rightarrow y = 1 - x + x^2 y$
- Natural implication (S-implication), see (Espin et al., 2011): $x \rightarrow y = d(n(x),y)$
- Zadeh implication (QL-implication): $x \rightarrow y = d(n(x),c(x,y))$
- Yager implication (A-implication): $x \rightarrow y = y^x$
- In this chapter, the implication operator with formula $x \rightarrow y = y^{2x}$ will be introduced and called *Generalized Yager implication*, which is a variation of the Yager implication. This operator will be introduced because of its satisfaction of the semantically intuitive property: $0.5 \rightarrow 0.5 = 0.5$

The formula of the equivalence is defined as: $x \leftrightarrow y = (x \rightarrow y) \wedge (y \rightarrow x)$. It is valid for any implication operator and any conjunction operator.

Other classifications can be found in (Jayaram, 2008).

Compensatory Inference Systems

Kleene's is a well-known Hilbert-style deduction system, therefore, the implication is an essential part of their logical axioms, which are tautologies with bivalent logic. They are defined as shown below (Dubois et al., 2007):

AX1: $A \rightarrow (B \rightarrow A)$
AX2: $(A \rightarrow B) \rightarrow ((A \rightarrow (B \rightarrow C)) \rightarrow (A \rightarrow C))$

AX3: $A \to (B \to A \wedge B)$

AX4: $A \wedge B \to A \quad A \wedge B \to B$

AX5: $A \to A \vee B \quad B \to A \vee B$

AX6: $(A \to C) \to ((B \to C)) \to (A \vee B \to C))$

AX7: $(A \to B) \to ((A \to \neg B) \to \neg A))$

AX8: $\neg(\neg A) \to A$

Here, the unique rule of inference is the modus ponens: "from A and $A \to B$, B is inferred".

A *proof* of the formula φ from the set of formulas Σ in this Hilbert system, is a finite sequence of formulas $\alpha_1, \alpha_2, \ldots, \alpha_n$, where $\alpha_n = \varphi$; such that each α_i is either a logical axiom, a member of Σ, or follows from an earlier α_j, α_k (j,k<i) by the modus ponens rule.

$\alpha_1, \alpha_2, \ldots, \alpha_n$ is a *formal deduction* of φ which is said to be *provable* in this Hilbert system, denoted by $\Sigma \vdash \varphi$. Σ is called the *hypotheses*.

An important metatheorem in this deductive system is the so-called *Deduction Theorem*, which consists of: if $\Sigma, \phi \vdash \varphi$, then $\Sigma \vdash \phi \to \varphi$.

A consequence of this metatheorem applied n times, is $\vdash \alpha_1 \to (\alpha_2 \to \ldots \to (\alpha_{n-1} \to \varphi)\ldots)$. Also, it is equivalent to $\vdash (\alpha_1 \wedge (\alpha_2 \wedge \ldots (\alpha_{n-2} \wedge \alpha_{n-1}) \ldots)) \to \varphi$.

These classical concepts of deductive systems and proofs can be naturally extended to fuzzy logic, using the CFL approach.

A Compensatory Inference System (CIS) is an octet $(L, Ax, R, c, d, o, n, \to)$, where c, d, o and n are the operators of a CFL, L is the set of formulas of the propositional calculus of the CFL, Ax is the Kleene's logical axioms, R is the modus ponens rule and $\to$ is the implication operator.

The definite integrals, double and triple, depending on which is the axiom, are calculated using MATLAB. The values of (11), for some fixed p and every implication operator are calculated and the results are summarized in the following tables:

	Natural Geometric	Generalized Zadeh	Yager	Reichenbach	Klir-Yuan	Generalized Yager
Ax1	0.5859	0.5685	0.8825	0.9143	0.7433	0.6065
Ax2	0.5122	0.5073	0.8425	0.8709	0.6745	0.6795
Ax3	0.5556	0.5669	0.8825	0.9088	0.7416	0.6065
Ax4	0.5859	0.5661	0.7436	0.8160	0.7148	0.5529
Ax5	0.5859	0.5859	0.7774	0.8160	0.7217	0.6044
Ax6	0.5026	0.5038	0.8772	0.8911	0.6617	0.6178
Ax7	0.5315	0.5137	0.7574	0.7882	0.6690	0.6624
Ax8	0.5981	0.5981	0.7788	0.8301	0.7413	0.6065
Univ. Quant.	0.5561	0.5502	0.8158	0.8532	0.7077	0.6160
Minimum	0.50258	0.5038	0.74357	0.78820	0.66172	0.5529

Table 1 Results of the evaluation of each Klenee's axiom in formula (11), where p=0 (Geometric mean). The ultimate two rows calculate the values of the Universal Quantifier and the Minimum, respectively, by implication operator.

	Natural	Generalized	Yager	Reichenbach	Klir-Yuan	Generalized

	Harmonic	Zadeh				Yager
Ax1	0.6369	0.6111	0.8784	0.9119	0.7369	0.5615
Ax2	0.5611	0.5449	0.8344	0.8668	0.6698	0.6488
Ax3	0.5996	0.6096	0.8686	0.9018	0.7335	0.5585
Ax4	0.6369	0.6161	0.7649	0.8295	0.7285	0.5699
Ax5	0.6369	0.6298	0.7905	0.8295	0.7265	0.6108
Ax6	0.5440	0.5331	0.8671	0.8823	0.6585	0.5703
Ax7	0.5753	0.5560	0.7411	0.7849	0.6640	0.6454
Ax8	0.6366	0.6366	0.7744	0.8270	0.7330	0.5934
Univ. Quant.	0.60116	0.58962	0.81181	0.85225	0.70475	0.5929
Minimum	0.544	0.5331	0.7411	0.78493	0.6585	0.5585

Table 2 Results of the evaluation of each Klenee's axiom in formula (11), where p=-1 (Harmonic mean).

		Generalized				Generalized
	Natural	Zadeh	Yager	Reichenbach	Klir-Yuan	Yager
Ax1	0.6527	0.6254	0.8743	0.9094	0.7308	0.5173
Ax2	0.5879	0.5677	0.826	0.8626	0.6656	0.6133
Ax3	0.6173	0.6242	0.858	0.8961	0.726	0.5185
Ax4	0.6527	0.6332	0.7715	0.8343	0.7311	0.5702
Ax5	0.6527	0.6413	0.7939	0.8343	0.7251	0.6041
Ax6	0.57	0.5565	0.86	0.8763	0.6554	0.5292
Ax7	0.5947	0.5782	0.7192	0.7819	0.6595	0.6277
Ax8	0.6446	0.6446	0.7703	0.824	0.7252	0.582
Univ. Quant.	0.6191	0.6061	0.8042	0.8496	0.69996	0.5658
Minimum	0.57	0.5565	0.7192	0.7819	0.6554	0.5173

Table 3 Results of the evaluation of each Klenee's axiom in formula (11), where p=-2.

		Generalized				Generalized
	Natural	Zadeh	Yager	Reichenbach	Klir-Yuan	Yager
Ax1	0.6552	0.6278	0.87	0.9068	0.7251	0.4777
Ax2	0.5991	0.5782	0.8174	0.8585	0.6619	0.5659
Ax3	0.6223	0.6268	0.8497	0.8913	0.719	0.4843
Ax4	0.6552	0.6367	0.7728	0.8355	0.7295	0.5654
Ax5	0.6552	0.6416	0.7936	0.8355	0.7215	0.5939
Ax6	0.5819	0.5688	0.8544	0.8717	0.6527	0.4937
Ax7	0.6014	0.5872	0.7033	0.779	0.6555	0.6084
Ax8	0.643	0.643	0.7665	0.8211	0.718	0.5722
Univ. Quant.	0.6242	0.61101	0.79627	0.84638	0.69482	0.5362
Minimum	0.5819	0.5688	0.7033	0.779	0.6527	0.4777

Table 4 Results of the evaluation of each Klenee's axiom in formula (11), where p=-3.

	Natural	Generalized	Yager	Reichenbach	Klir-Yuan	Generalized

		Zadeh				Yager
Ax1	0.6521	0.6252	0.8656	0.9042	0.7198	0.4447
Ax2	0.6026	0.5821	0.8085	0.8543	0.6586	0.4594
Ax3	0.6213	0.6243	0.8428	0.8871	0.7125	0.4549
Ax4	0.6521	0.6345	0.7719	0.835	0.7262	0.5595
Ax5	0.6521	0.6376	0.7915	0.835	0.7173	0.5835
Ax6	0.5864	0.5741	0.8495	0.8679	0.6503	0.4636
Ax7	0.6021	0.5899	0.6331	0.7763	0.6521	0.5864
Ax8	0.6379	0.6379	0.7629	0.8184	0.7114	0.5639
Univ. Quant.	0.6233	0.6106	0.77164	0.84291	0.6899	0.4979
Minimum	0.5864	0.5741	0.6331	0.7763	0.6503	0.4447

Table 5 Results of the evaluation of each Klenee's axiom in formula (11), where p=-4.

	Natural	Generalized Zadeh	Yager	Reichenbach	Klir-Yuan	Generalized Yager
Ax1	0.6142	0.5941	0.8395	0.8876	0.6947	0.3452
Ax2	0.5859	0.5711	0.7525	0.8309	0.6449	0
Ax3	0.5935	0.5936	0.8149	0.8673	0.6833	0.3541
Ax4	0.6142	0.6018	0.7565	0.8231	0.7016	0.5324
Ax5	0.6142	0.6022	0.7715	0.8231	0.6929	0.5423
Ax6	0.5751	0.5665	0.8272	0.8519	0.6401	0.3609
Ax7	0.5817	0.5755	0.3451	0.7635	0.6378	0
Ax8	0.601	0.601	0.7462	0.8048	0.683	0.5341
Univ. Quant.	0.5954	0.58637	0.4248	0.8225	0.66699	0
Minimum	0.5751	0.5665	0.3451	0.7635	0.6378	0

Table 6 Results of the evaluation of each Klenee's axiom in formula (11), where p=-10.

	Natural	Generalized Zadeh	Yager	Reichenbach	Klir-Yuan	Generalized Yager
Ax1	0.5094	0.5085	0.7089	0.7677	0.6183	0.2367
Ax2	0.5084	0.5077	0	0.7228	0.6025	0
Ax3	0.5085	0.5085	0	0.7666	0.6036	0
Ax4	0.5094	0.5089	0.6983	0.7572	0.6222	0.4835
Ax5	0.5094	0.5089	0.6991	0.7572	0.6216	0.4838
Ax6	0.5079	0.5076	0.6925	0.7502	0.6024	0.2079
Ax7	0.5081	0.5079	0	0.713	0.5993	0
Ax8	0.5088	0.5088	0.6978	0.7561	0.6209	0.4836
Univ. Quant.	0.5086	0.5083	0	0.7179	0.6026	0
Minimum	0.5079	0.5076	0	0.713	0.5993	0

Table 7 Results of the evaluation of each Klenee's axiom in formula (11), where p=-300.

The tables show that the Reichenbach implication has the best minimum values for all of the fixed p. Axiom 7 has the worst truth-values. Axioms like the fourth and fifth increase their truth-values if p decreases, until some p is reached and they tend to decrease. However, generally the truth-values decrease if p decreases.

Because the best results were obtained with Reichenbach, Yager and Klir-Yuan implications, they were selected as candidates for defining the CIS. It is an optimization problem, where p and the implication operators are the parameters to estimate, and the eight objective functions to maximize are the Kleene's axioms evaluated in formula (11) of validation. Genetic algorithm of MATLAB was used for the optimization. The results are shown in table 8.

Implication/Axiom	Reichenbach implication		Yager implication		Klir-Yuan implication	
	p estimated	Truth-value	p estimated	Truth-value	p estimated	Truth-value
Ax1	0	0.91431701	-1.907e-6	0.88259685	0	0.74328787
Ax2	0	0.87092980	-1.9073e-6	0.85457610	0	0.67454376
Ax3	-1.9073e-6	0.90879967	0	0.88251055	0	0.74160008
Ax4	-3.13379	0.83549684	-2.95215	0.77283009	-1.97656	0.73109926
Ax5	-3.13379	0.83549684	-2.33789	0.79407098	-1.09863	0.72658054
Ax6	0	0.89184635	0	0,87716883	0	0,66901104
Ax7	0	0.78820226	0	0.75737807	-1.9073e-6	0.66901135
Ax8	-1.9073e-6	0.83010704	-1.9073e-6	0.77897721	0	0.74125505

Table 8 Results of maximizing the validation function evaluated in every Kleene's axiom, by the three selected implication operators. The parameter p was estimated using Genetic algorithm of MATLAB.

Table 8 confirms preliminary results in the above tables; Reichenbach implication generates the biggest truth-values for each Kleene's axiom. The best estimated p were p=0 or -1.9073e-6, except for the fourth and fifth axioms, where p is between -1 and -4. The seventh axiom is the worst in all cases, except in Klir-Yuan implication. Because the minimum of the truth-values in Reichenbach implication is reached in the seventh axiom, and the p estimated for this is 0, then the Compensatory Logic on Geometric Mean is selected to form part of the CIS, associated with the Reichenbach implication.

That is to say, → is the Reichenbach implication in the octet of the CIS, while *c* and *d* operators are defined, respectively, as the geometric mean and its dual.

The truth-value of the below formula (12) could be taken as the truth-value of this demonstration of φ.

$$(\alpha_1 \wedge (\alpha_2 \wedge \ldots (\alpha_{n-2} \wedge \alpha_{n-1}) \ \ldots)) \rightarrow \varphi \quad (12)$$

Let us note that due CFL is not associative; parentheses in (12) can't be eliminated.

From the point of view of Decision Theory (French, 1986), it is a normative (rational) approach to fuzzy deduction. The actual way for deducting by persons is not reflected in this chapter, despite the fact that the use of compensatory operators is closer to this way.

If parameter p in (2) measures the degree of 'pessimism' in decision making, it may be said that the CIS is a neuter manner, and hence rational way for making a decision. Since formula (2) becomes in the minimum and maximum's operators respectively when p= -∞ or p = +∞, (Mitrinovic et al., 1993; Detyniecki, 2001), then p=0 is the most 'compensatory' of all the compensatory systems in the one-parameter family exposed in (2).

Example

An illustrative example of deduction in bivalent logic with the Kleene's axioms is:

$A, B, A\rightarrow(B\rightarrow(A\wedge B)), B\rightarrow A\wedge B \vdash A\wedge B$

A and B are formulas of the hypothesis, $A\rightarrow(B\rightarrow(A\wedge B))$ is axiom 3, $B\rightarrow A\wedge B$ is obtained from $A\rightarrow(B\rightarrow(A\wedge B))$ and A applying modus ponens, and $A\wedge B$ is inferred from B and $B\rightarrow A\wedge B$ applying modus ponens.

The truth-value of this proof calculated with formula (12), using the CIS, is approximately 0.76.

It is important to remark that due to the properties of idempotency and 'sensitiveness' of the CFL, and its Decision theory origin, the truth-values of composed predicates can be considered utility functions and they can be interpreted as cardinal decision functions according to the Measurement theory of Decision theory (French, 1986). Hence, the truth-values of the composed predicates can be interpreted semantically by themselves. It is an advantage of the CFL over the t-norm and t-conorm systems. Even an equivalent categorical correspondence is used to link a numerical truth-value with categories, see table 10. Hence, according to table 10, the 'validity' of the proof in the example is 'enough true'.

Truth-value	Category
0	Absolutely false
0.1	Almost false
0.2	Enough false
0.3	Somewhat false
0.4	More false than true
0.5	As true as false
0.6	More true than false
0.7	Somewhat true
0.8	Enough true
0.9	Almost true
1	Absolutely true

Table 9 Categorical table of truth-values

Concluding Remarks

This chapter introduced a new fuzzy inference system based on Compensatory Fuzzy Logic. It is an octet $(L,Ax,R,c,d,o,n,\rightarrow)$, where *c*, *d*, *o* and *n* are the operators of the Compensatory Logic on Geometric Mean, L is the set of formulas of the propositional calculus in this CFL, Ax is the Kleene's logical axioms, R is the modus ponens rule and $\rightarrow$ is the Reichenbach implication. This combination maximizes the validity of the Kleene's axioms, as it was exposed above.
Compensatory Inference System, as it was named, is a natural extension of deduction in Classical logic with easy procedures for computing. Hence, it is an inference system with easy application and logically rigorous. Therefore, it satisfies at the same time the main purpose of Fuzzy logic in narrow and in broad senses, that is to say, there exists a natural link with Mathematical logic and it is applicable.
Some recommendations, which shall be further themes of research by authors, are the inclusion of other implications in this study, like for example f- and g-implications. The set of axioms can be extended to predicate calculus with the universal and existential quantifiers, and others of the non-classical logic. Authors are making a parallel study of deduction with a descriptive (real) approach, in contrast to the normative development made in this chapter.

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