Weak Sharp Minima and Finite Termination of the Proximal Point Method for Convex Functions on Hadamard Manifolds

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Abstract

In this work is showed that the sequence generated by the proximal point method, associated to a unconstrained optimization problem in the Riemannian context, has finite termination when the objective function has a weak sharp minima on the solution set of the problem.

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Main area: Mathematical programming.

1 Introdução

Consider the following minimization problem

$$(P) \quad \begin{array}{l} \min f(p) \\ \text{s.t. } \; p \in M, \end{array} \tag{1}$$

where M is a complete Riemannian manifold and $f : M \to \mathbb{R}$ is a function. For a starting point $p^0 \in M$, the exact proximal point method to solve optimization problems of the form (1) generates a sequence $\{p^k\} \subset M$ as follows:

$$p^{k+1} \in \text{argmin}_{p\in M} \left\{ f(p) + \lambda_k d^2(p, p^k) \right\}, \tag{2}$$

where $\{\lambda_k\}$ is a sequence of positive numbers and d is the Riemannian distance (see Section 2 for a definition). This method was first considered in this context by Ferreira and Oliveira (2002), when M is a Hadamard manifold (see Section 2 for a definition) and f is convex. It is worth noting that Li et al. (2009) extended this method for finding singularity of a multivalued vector field and proved that the generated sequence is well-defined and converges to a singularity of a maximal monotone vector field, whenever it exists.

Assuming that the sequence $\{p^k\}$ generated by (2) is well defined, it follows that

$$(\lambda_k/2)d^2(p^{k+1}, p^k) \leq f(p^k) - f(p^{k+1}).$$

Hence, if f is bounded below, then

$$\sum_{k=0}^{+\infty} (\lambda_k/2)d^2(p^{k+1}, p^k) < +\infty. \tag{3}$$

The following factos gathers some of the main results of Ferreira and Oliveira (2002) associated to the sequence $\{p^k\}$.

Proposition 1.0.1 *If M is a Hadamard manifold, then the following statements hold:*

i) $\{p^k\}$ is well defined and is characterized by

$$\lambda_k(exp^{-1}_{p_{k+1}} p_k) \in \partial f(p_{k+1}); \tag{4}$$

ii) If $\sum_{k=0}^{+\infty} 1/\lambda_k = +\infty$ and the solution set of the problem (1) is nonempty, then the sequence $\{p^k\}$ converges to a solution of the problem (1).

Following the ideas of Ferris (1991), we showed in this work that the sequence generated by the proximal point method associated to the problem (1) has finite termination when the objective function is convex, M is a Hadamard manifold and the solution set of the Problem 1, denoted by U, is a set of weak sharp minimizers for f, see Section 3 for a definition. As far as we know, the notion of sharp minimizer was introduced by Polyak (1979) for the case of finite-dimensional Euclidean spaces; see also page 205 in Polyak (1987). In this particular case it is know that a necessary and sufficient condition for $\bar{p}$ be sharp minimum is that $0 \in \text{int}\partial f(\bar{p})$. Rockafellar (1976) showed that, in a space with linear structure (Hilbert space), this is a sufficient condition for finite termination of the proximal point method. Afterwards, Burke and Ferris (1993) extended the notion of sharp minima to what became known as weak sharp minima, mainly to include the possibility of multiple solutions, and extended the previous necessary and sufficient condition for characterize the solution set of a minimization problem as a set of weak sharp minimizers. Li et al. (2011) extended the notion of weak sharp minimizer to optimization problems on Riemannian manifolds as well as the previous result which relates finite termination of the proximal point method with weak sharp minima, summarized as follows:

"A necessary and sufficient condition for U be the set of weak sharp minima for Problem 1 with modulus $\alpha > 0$ is that

$$\alpha \mathcal{B}_p \cap N_U(p) \subset \partial f(p), \qquad p \in U,$$

where $\mathcal{B}_p$ is the closed unit ball of T_pM and $N_U(p)$ is the normal cone to U at p (see Section 2 for definitions)".

The organization of our work is as follows. In Section 2, some notations and results of Riemannian geometry as well as some fundamental properties and notations of convex analysis on Hadamard manifolds, are presented. In Section 3, it is presented the definition of weak sharp minima as well as the main resulted of this work. Finally, in Section 4, we made some last considerations.

2 Notation and terminology

In this section we introduce some fundamental properties and notations on Riemannian geometry and convex analysis on Hadamard manifolds. For references on Riemannian geometry, see any introductory book on Riemannian geometry, such as in do Carmo (1992) and Sakai (1996). Now, for references on convex analysis, see Udriste (1994), Rapcsák (1997) and Li et al. (2009).

Let M be a n-dimentional connected manifold. We denote by T_pM the n-dimentional *tangent space* of M at p, by $TM = \cup_{p \in M} T_pM$ *tangent bundle* of M and by $\mathcal{X}(M)$ the space of smooth vector fields over M. When M is endowed with a Riemannian metric $\langle\, , \,\rangle$, with the corresponding norm denoted by $\|\ \|$, then M is now a Riemannian manifold. We denote by $\mathcal{B}_p := \{v \in T_pM : \|v\| \leq 1\}$ the closed unit ball of T_pM. Recall that the metric can be used to define the lenght of piecewise smooth curves $\gamma : [a, b] \to M$ joining p to q, i.e., such that $\gamma(a) = p$ and $\gamma(b) = q$, by

$$l(\gamma) = \int_a^b \|\gamma'(t)\| dt,$$

and, moreover, by minimizing this length functional over the set of all such curves, we obtain a Riemannian distance $d(p, q)$ which induces the original topology on M. Given a nonempty set $D \subset M$, the distance function associated with D is given by

$$M \ni p \longmapsto d_D(p) := \inf\{d(q, p) : q \in M\} \in \mathbb{R}_+.$$

The metric induces a map $f \mapsto \text{grad} f \in \mathcal{X}(M)$ which associates to each smooth function on M its gradient via the rule $\langle \text{grad} f, X\rangle = df(X),\ X \in \mathcal{X}(M)$. Let ∇ be the Levi-Civita connection associated to $(M, \langle\, , \,\rangle)$. A vector field V along γ is said to be *parallel* if $\nabla_{\gamma'} V = 0$. If γ' itself is parallel we say that γ is a *geodesic*. Given that geodesic equation $\nabla_{\ \gamma'}\gamma' = 0$ is a second order nonlinear ordinary differential equation, then geodesic $\gamma = \gamma_v(., p)$ is determined by its position p and velocity v at p. It is easy to check that $\|\gamma'\|$ is constant. We say that γ is *normalized* if $\|\gamma'\| = 1$. The restriction of a geodesic to a closed bounded interval is called a *geodesic segment*. A geodesic segment joining p to q in M is said to be *minimal* if its length equals $d(p, q)$ and this geodesic is called a *minimizing geodesic*. A Riemannian manifold is *complete* if geodesics are defined for any values of t. Hopf-Rinow's theorem ([10, Theorem 1.1, page 84]) asserts that if this is the case then any pair of points, say p and q, in M can be joined by a (not necessarily unique) minimal geodesic segment. Moreover, (M, d) is a complete metric space and bounded and closed subsets are compact. Take $p \in M$. The *exponential map* $exp_p : T_pM \to M$ is defined by $exp_p v = \gamma_v(1, p)$.

Theorem 2.0.1 *Let M be a complete, simply connected Riemannian manifold with nonpositive sectional curvature. Then M is diffeomorphic to the Euclidean space $\mathbb{R}^n$, $n = dim M$. More precisely, at any point $p \in M$, the exponential mapping $exp_p : T_pM \to M$ is a diffeomorphism.*

Proof: See Lemma 3.2 of do Carmo (1992), page 149 or Theorem 4.1 of Sakai (1996), page 221. ■

A complete simply connected Riemannian manifold of nonpositive sectional curvature is called a *Hadamard manifold.* The Theorem 2.0.1 says that if M is Hadamard manifold, then M has the same topology and differential structure of the Euclidean space $\mathbb{R}^n$. Furthermore, some similar geometrical properties of the Euclidean space $\mathbb{R}^n$ are known, such as, given two points there exists an unique geodesic that joins them. *In this paper, all manifolds M are assumed to be Hadamard finite dimensional.*

Take $p \in M$. Let $exp_p^{-1} : M \to T_pM$ be the inverse of the exponential map which is also C^∞. Note that $d(q\,,\,p) = ||exp_p^{-1}q||$, the map $d^2(\,.\,,p)\colon M \to \mathbb{R}$ is C^∞ and

$$\text{grad}\frac{1}{2}d^2(q,p) = -exp_q^{-1}p,$$

see, for example, Proposition 4.8 of Sakai (1996), page 108.

A set $\Omega \subset M$ is said to be *convex* if any geodesic segment with end points in Ω is contained in Ω. A function $f : M \to \mathbb{R}$ is said to be *convex* if for any geodesic segment $\gamma : [a,b] \to M$ the composition $f \circ \gamma : [a,b] \to \mathbb{R}$ is convex. Take $p \in M$. A vector $s \in T_pM$ is said to be a *subgradient* of f at p if

$$f(q) \geq f(p) + \langle s, \exp_p^{-1} q\rangle,$$

for any $q \in M$. The set of all subgradients of f at p, denoted by $\partial f(p)$, is called the *subdifferential* of f at p. It is known that if f is convex then $\partial f(p)$ is a set non-empty, convex and compact, for each $p \in M$.

Let $D \subset \mathbb{R}^n$ be a convex set, and $p \in D$. Following Li et al. (2009), we define the normal cone to D at p by:

$$N_D(p) := \{w \in T_pM : \langle w, \exp_p^{-1} q\rangle \leq 0, q \in D\}.$$

The previous definition holds just when M is of the Hadamard type. A more general definition has appeared in Li et al. (2011).

3 Proximal Point and Weak Sharp Minima on Riemannian Manifolds

In this section we present the definition of weak minima sharp in the Riemannian context as well as the main resulted of this work, more accurately, that the proximal point method (2) has finite termination when the solution set of Problem (1) (nonempty and closed) is a set of weak sharp minimizers.

Definition 3.0.1 *The set U is said be the set of weak sharp minimizers for Problem 1 with modulus $\alpha > 0$ if*

$$f(q) \geq f(p) + \alpha d_U(q), \qquad p \in U, \quad q \in M.$$

Next we present the central result of this paper.

Theorem 3.0.2 *Suppose that U is the set of weak sharp minima for Problem 1 with modulus $\alpha > 0$ and let $p^0 \in \mathbb{R}^n$. If $\{\lambda_k\}$ is a sequence of real numbers and λ_-, λ_+ positive constants such that $\lambda_- \leq \lambda_k \leq \lambda_+$, $k \in \mathbb{N}$, then the proximal point method terminates in a finite number of iterations.*

Proof: See Bento and Cruz Neto (2012). ■

4 Final Remarks

In this work we recall the notion of weak sharp minima for unconstrained optimization problem on Riemannian manifolds and we explored properties of weak sharp minimum on Hadamard manifold to establish the finite termination of the proximal point method.

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