F-TODIM: AN APPLICATION OF THE FUZZY TODIM METHOD TO RENTAL EVALUATION OF RESIDENTIAL PROPERTIES

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RESUMO

O método de *T*omada de *D*ecisão *I*terativa *M*ulticritério (TODIM) é um dos primeiros métodos de tomada decisão baseado na teoria da propensão ao risco. Um característica marcante deste método é sua capacidade para tratar problemas de tomada de decisão envolvendo riscos. Contudo, o método TODIM na sua formulação original não é capaz de levar em conta informações incertas da matriz de decisão. Objetivando atacar esse problema, foi desenvolvido recentemente pelos autores desse artigo o método *Fuzzy* TODIM, que é uma extensão do método TODIM. Dessa forma torna-se possível tratar problemas de tomada de decisão envolvendo riscos e incertezas. Um estudo de caso ilustrando a aplicação na avaliação de imóveis residenciais para alugar é apresentado.

Palavras chave. Tomada de decisão multi-critério, teoria da propensão ao risco, *fuzzy* TODIM

ADM – Apoio à Decisão Multicritério

ABSTRACT

The TODIM method, which is an acronym in Portuguese for Iterative Multi-criteria Decision Making, is one of the first Multi-Criteria Decision Making (MCDM) founded on prospect Theory. One of the strong attributes is its capacity to treat risk MCDM problems. Nevertheless, TODIM in its original formulation is not able to take into account uncertain information of the decision matrix. In order to tackle this shortcoming, the authors of this paper have recently developed the Fuzzy TODIM method, which is an extension of TODIM method to handle uncertain MCDM problems. So, it is possible to handle risk and uncertain MCDM problems. A case study illustrating the application of the method to rental evaluation of residential properties is presented.

Keywords. Multi-Criteria Decision Making (MCDM), prospect theory, fuzzy TODIM

MCDM – Multi-criteria Decision Making

1. Introduction

Complex decision processes may be considered difficult to solve most due to the involved uncertainties, associated risks and inherent complexities of multi-criteria decision making (MCDM) problems (Fenton and Wang, 2006). The theory of fuzzy sets and fuzzy logic developed by Zadeh (1965) has been used to model uncertainty or lack of knowledge and applied to several MCDM problems. Bellman & Zadeh (1970) introduced the theory of fuzzy sets in problems of MCDM as an effective approach to treat vagueness, lack of knowledge and ambiguity inherent in the human decision making process which are known as fuzzy multi-criteria decision making (FMCDM). See for example Zimmerman (1991) for more information.

For real world-problems the decision matrix is affected by uncertainty and may be modeled using fuzzy numbers. A fuzzy number (Dubois and Prade, 1980) can be seen as an extension of an interval with varied grade of membership. This means that each value in the interval has associated a real number that indicates its compatibility with the vague statement associated with a fuzzy number. So, standard MCDM methods like TOPSIS (Wang and Yoon, 1981) and PROMETHEE (Brans, Vincke and Marechal, 1986) have been extended using fuzzy numbers resulting in fuzzy TOPSIS (Wang, Liu, and Zhang, 2005) and fuzzy PROMETHEE (Goumas and Lygerou, 2000), respectively. Both methods have successfully been applied to solve various uncertain MCDM problems.

Another important aspect of decision making is that most of the existing MCDM methods are not able to capture or take into account the risk attitude/preferences of the decision maker in MCDM. Prospect theory developed by Kahneman and Tversky (1979) is a descriptive model of individual decision making under condition of risk. Later, Tversky and Kahneman (1992) developed the cumulative prospect theory, which capture psychological aspects of decision making under risk. In the prospect theory, the outcomes are expressed by means of gains and losses from a reference alternative (Salminen, 1994). The value function in prospect theory assumes an *S*-shape concave above the reference alternative, which reflects the aversion of risk in face of gains; and the convex part below the reference alternative reflects the propensity to risk in case of losses.

As far as we know, one of the first MCDM methods based on prospect theory was proposed by Gomes and Lima (1992). Despite its effectiveness and simplicity in concept, this method presents some shortcomings because of its inability to deal with uncertainty and imprecision inherent in the process of decision making. In the original formulation of TODIM (an acronym in Portuguese for Iterative Multi-criteria Decision Making), the rating of alternatives, which composes the decision matrix, is represented by crisp values. The TODIM method has many similarities with the PROMETHEE method, whereas the preference function is replaced by the prospect function. The TODIM method has been applied to rental evaluation of residential properties (Gomes and Rangel, 2009) among others applications with good performance. However, one of the shortcomings of the TODIM method is its inability to handle uncertain MCDM, which are present in many MCDM problems.

In a previous work (Krohling & de Souza, 2012), we extend the TODIM method by combining the strong aspects of prospect theory and fuzzy sets to handle uncertain and risk MCDM and developed the Fuzzy TODIM method, which is able to handle uncertain decision matrices. In section 2, we provide some background knowledge on fuzzy sets and prospect theory. In section 3, we present the novel Fuzzy TODIM method, which contains uncertainty in the decision matrix using trapezoidal fuzzy numbers and the value function of prospect theory to handle risk attitudes of the decision maker. In section 4, we present a case study to illustrate the method. In section 5, we present some conclusions and directions for future work.

2. Multi-criteria Decision Making

2.1 Preliminaries on prospect theory

The value function used in the prospect theory is described in form of a power law according to the following expression:

$$v(x)=\begin{cases} x^{\alpha} & \text{if } x\ge 0 \\ -\theta(-x)^{\beta} & \text{if } x<0 \end{cases} \quad (1)$$

where α and β are parameters related to gains and losses, respectively. The parameter θ represents a characteristic of being steeper for losses than for gains. In case of risk aversion $\theta > 1$. Fig. 1 shows a prospect value function with a concave and convex S-shaped for gains and losses, respectively. Kahneman and Tversky (1979) experimentally determined the values of $\alpha = \beta = 0.88$, and $\theta = 2.25$, which are consistent with empirical data. Further, they suggested that the value of θ is between 2.0 and 2.5.

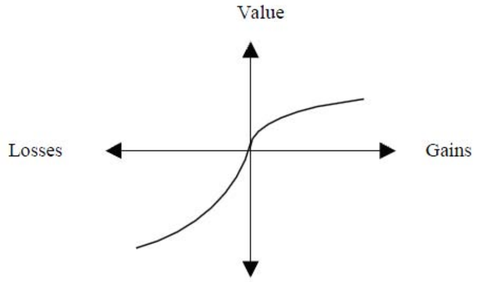

Fig. 1: Value function of prospect theory.

2.2 Preliminaries on fuzzy sets and fuzzy number

Next, we provide some basic definitions of fuzzy sets and fuzzy numbers (Wang & Lee, 2009).

Definition 1: A fuzzy set $\tilde{A}$ in a universe of discourse X is characterized by a membership function $\mu_{\tilde{A}}(x)$ that assigns each element x in X a real number in the interval [0; 1]. The numeric value $\mu_{\tilde{A}}(x)$ stands for the grade of membership of x in $\tilde{A}$.

Definition 2: A trapezoidal fuzzy number $\tilde{a}$ is defined by a quadruplet $\tilde{a}=(a_1,a_2,a_3,a_4)$ as shown in Fig. 2. The membership function is given by:

$$\mu_{\tilde{a}}(x)=\begin{cases} 0, & \text{if } x<a_1 \\ \dfrac{x-a_1}{a_2-a_1}, & \text{if } a_1 \le x < a_2 \\ 1, & \text{if } a_2 \le x \le a_3 \\ \dfrac{a_4-x}{a_4-a_3}, & \text{if } a_3 < x \le a_4 \\ 0, & \text{if } x > a_4 \end{cases} \quad (2)$$

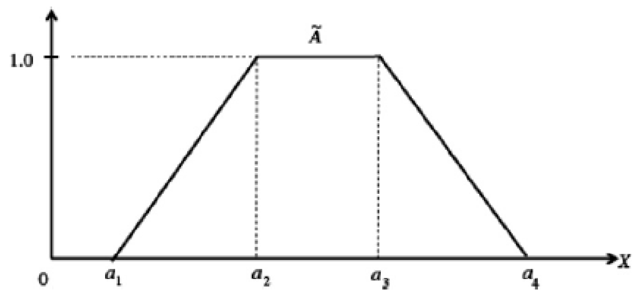

Fig. 2: Trapezoidal fuzzy number $\tilde{a}=(a_1,a_2,a_3,a_4)$.

Definition 2: Let a trapezoidal fuzzy number $\tilde{a} = (a_1, a_2, a_3, a_4)$, then the defuzzified value $m_{\tilde{a}}$ is calculated by:

$$m_{\tilde{a}} = \frac{(a_1 + a_2 + a_3 + a_4)}{4}. \quad (3)$$

Definition 3: Let two trapezoidal fuzzy numbers $\tilde{a} = (a_1, a_2, a_3, a_4)$ and $\tilde{b} = (b_1, b_2, b_3, b_4)$, then the operation with these fuzzy numbers are defined as follows:

1. Addition of fuzzy numbers (+)
$\tilde{a}(+)\tilde{b} = (a_1, a_2, a_3, a_4)\ (+)\ (b_1, b_2, b_3, b_4) = (a_1 + b_1,\ a_2 + b_2,\ a_3 + b_3, a_4 + b_4)$
2. Subtraction of fuzzy numbers (-)
$\tilde{a}(-)\tilde{b} = (a_1, a_2, a_3, a_4)\ (-)\ (b_1, b_2, b_3, b_4) = (a_1 - b_1,\ a_2 - b_2,\ a_3 - b_3,\ a_4 - b_4)$
3. Multiplication of fuzzy numbers (*x*)
$\tilde{a}(x)\tilde{b} = (a_1, a_2, a_3, a_4)\ (x)\ (b_1, b_2, b_3, b_4) = (a_1 \cdot b_1,\ a_2 \cdot b_2,\ a_3 \cdot b_3,\ a_4 \cdot b_4)$
4. Division of fuzzy numbers (/)
$\tilde{a}(/)\tilde{b} = (a_1, a_2, a_3, a_4)\ (/)\ (b_1, b_2, b_3, b_4) = (a_1 / b_4,\ a_2 / b_3,\ a_3 / b_2,\ a_4 / b_1)$
5. Multiplication by a scalar number *k*
$k\tilde{a} = k(a_1, a_2, a_3, a_4) = (ka_1,\ ka_2,\ ka_3, ka_4)$.

Definition 4: Let two trapezoidal fuzzy numbers $\tilde{a} = (a_1, a_2, a_3, a_4)$ and $\tilde{b} = (b_1, b_2, b_3, b_4)$, then the distance between them (Mahdavi *et al.,* 2008) is calculated by:

$$d(\tilde{a}, \tilde{b}) = \sqrt{\frac{1}{6}\left[\sum_{i=1}^{4}(b_i - a_i)^2 + \sum_{i \in \{1,3\}} (b_i - a_i)(b_{i+1} - a_{i+1})\right]}. \quad (4)$$

Definition 5: Let the trapezoidal fuzzy number $\tilde{a} = (a_1, a_2, a_3, a_4)$, then the following properties of prospect value of fuzzy numbers are given as follows:
(1) if $a_1 \ge 0, a_2 \ge 0, a_3 \ge 0, a_4 \ge 0,$ then $v(\tilde{a}) = [a_1{}^{\alpha}, a_2{}^{\alpha}, a_3{}^{\alpha}, a_4{}^{\alpha}]$
(2) if $a_1 \le 0, a_2 \le 0, a_3 \le 0, a_4 \le 0$ then $v(\tilde{a}) = [-\theta a_1{}^{\beta}, -\theta a_2{}^{\beta}, -\theta a_3{}^{\beta}, -\theta a_4{}^{\beta}]$
(3) if $a_1 \le 0, a_2 \le 0, a_3 \ge 0, a_4 \ge 0,$ then $v(\tilde{a}) = [-\theta a_1{}^{\beta}, -\theta a_1{}^{\beta}, a_3{}^{\alpha}, a_4{}^{\alpha}]$
(4) if $a_1 \le 0, a_2 \ge 0, a_3 \ge 0, a_4 \ge 0,$ then $v(\tilde{a}) = [-\theta a_1{}^{\beta}, a_2{}^{\alpha}, a_3{}^{\alpha}, a_4{}^{\alpha}]$
(5) if $a_1 \le 0, a_2 \le 0, a_3 \le 0, a_4 \ge 0$ then $v(\tilde{a}) = [-\theta a_1{}^{\beta}, -\theta a_2{}^{\beta}, -\theta a_3{}^{\beta}, a_4{}^{\alpha}]$
with $\alpha \in [0,1], \beta \in [0,1], \theta > 1$.

3 Fuzzy Multicriteria Decision Making

3.1 Description of decision making problem with uncertain decision matrix

Let us consider the fuzzy decision matrix *A,* which consists of a*lternatives* and *criteria,* described by:

$$\begin{array}{cc} & \begin{array}{ccc} C_1 & \dots & C_n \end{array} \\ \tilde{\underset{\cdot\cdot}{A}} = \begin{array}{c} A_1 \\ \dots \\ A_m \end{array} & \begin{pmatrix} \tilde{x}_{11} & \cdots & \tilde{x}_{1n} \\ \vdots & \ddots & \vdots \\ \tilde{x}_{m1} & \cdots & \tilde{x}_{mn} \end{pmatrix} \end{array}$$

where $A_1, A_2, \cdots, A_m$ are alternatives, $C_1, C_2, ..., C_n$ are criteria, $\tilde{x}_{ij}$ are fuzzy numbers that indicates the rating of the alternative A_i with respect to criterion C_j. The weight vector

$W = (w_1, w_2 ..., w_n)$ composed of the individual weights $w_j\,(j = 1,...,n)$ for each criterion C_j satisfying $\sum_{i=1}^{n} w_j = 1$. The data of the decision matrix A come from different sources, so it is necessary to normalize it in order to transform it into a dimensionless matrix, which allow the comparison of the various criteria. In this work, we use the normalized decision matrix $R = \left[r_{ij} \right]_{mxn}$ with $i = 1,...,m,$ and $j = 1,...,n.$

After normalizing the decision matrix and the weight vector, TODIM begins with the calculation of the partial dominance matrices and the final dominance matrix. For such calculations the decision makers need to define firstly a reference criterion, which usually is the criterion with the highest importance weight. So, w_{rc} indicates the weight of the criterion c divided by the reference criteria r. TODIM is described in (Gomes and Lima, 1992; Gomes & Rangel, 2009).

The use of numerical values in the rating of alternatives may have limitations to deal with uncertainties. So, an extension of TODIM is developed to solve problems of decision making with uncertain data resulting in fuzzy TODIM. In practical applications, the trapezoidal shape of the membership function is often used to represent fuzzy numbers. Fuzzy models using trapezoidal fuzzy numbers proved to be very effective for solving decision-making problems where the available information is imprecise.
The fuzzy TODIM is described in the following.

3.2 The Fuzzy TODIM method

The Fuzzy TODIM is described in the following steps:

Step 1: The criteria are normally classified into two types: *benefit* and *cost*. The fuzzy-decision matrix $\tilde{A} = \left[\tilde{x}_{ij} \right]_{mxn}$ with $i = 1,...,m,$ and $j = 1,...,n$ is normalized which results the correspondent fuzzy-decision matrix $\tilde{R} = \left[\tilde{r}_{ij} \right]_{mxn}$. The fuzzy normalized value $\tilde{r}_{ij}$ is calculated as:

$$r_{ij}{}^{k} = \frac{\max(a_{ij}{}^{4}) - a_{ij}{}^{k}}{\max_i (a_{ij}{}^{4}) - \min_i a_{ij}{}^{1}} \quad \text{with } k=1,2,3,4 \qquad \text{for cost criteria}$$

$$r_{ij}{}^{k} = \frac{a_{ij}{}^{k} - \min(a_{ij}{}^{1})}{\max_i (a_{ij}{}^{4}) - \min_i a_{ij}{}^{1}} \quad \text{with } k=1,2,3,4 \qquad \text{for benefit criteria} \tag{5}$$

Step 2: Calculate the dominance of each alternative $\tilde{A}_i$ over each alternative $\tilde{A}_j$ using the following expression:

$$\delta(\tilde{A}_i, \tilde{A}_j) = \sum_{c=1}^{m} \phi_c(\tilde{A}_i, \tilde{A}_j) \qquad \forall (i, j) \tag{6}$$

where

$$\phi_c(\tilde{A}_i, \tilde{A}_j) = \begin{cases} \sqrt{\dfrac{w_{rc}}{\sum_{c=1}^{m} w_{rc}}} \cdot d(\tilde{x}_{ic}, \tilde{x}_{jc}) & \text{if } [m_{(\tilde{x}_{ic})} - m(_{\tilde{x}_{jc}})] > 0 \\ 0, & \text{if } [m_{(\tilde{x}_{ic})} - m(_{\tilde{x}_{jc}})] = 0 \\ \dfrac{-1}{\theta} \sqrt{\dfrac{\left(\sum_{c=1}^{m} w_{rc}\right)}{w_{rc}}} \cdot d(\tilde{x}_{ic}, \tilde{x}_{jc}) & \text{if } [m_{(\tilde{x}_{ic})} - m(_{\tilde{x}_{jc}})] < 0 \end{cases} \tag{7}$$

The term $\phi_c(\tilde{A}_i, \tilde{A}_j)$ represents the contribution of the criterion c to the function $\delta(\tilde{A}_i, \tilde{A}_j)$ when comparing the alternative i with alternative j. The parameter θ represents the attenuation factor of the losses, which can be tuned according to the problem at hand. In Equation (7) $m_{(\tilde{x}_{ic})}$ and $m_{(\tilde{x}_{jc})}$ stands for the defuzzified values of the fuzzy number $\tilde{x}_{ic}$ and $\tilde{x}_{jc}$, respectively. The term $d(\tilde{x}_{ic}, \tilde{x}_{jc})$ designates the distance between the two fuzzy numbers $\tilde{x}_{ic}$ and $\tilde{x}_{jc}$, as defined in Equation (4). Three cases can occur in Equation (7): i) if the value $m_{(\tilde{x}_{ic})} - m(\tilde{x}_{jc})$ is positive, it represents a gain; ii) if the value $m_{(\tilde{x}_{ic})} - m(\tilde{x}_{jc})$ is nil; and iii) if the value $m_{(\tilde{x}_{ic})} - m(\tilde{x}_{jc})$ is negative, it represents a loss. The final matrix of dominance is obtained by summing up the partial matrices of dominance for each criterion.

Step 3: Calculate the global value of the alternative i by means of normalizing the final matrix of dominance according to the following expression:

$$\xi_i = \frac{\sum \delta(i,j) - \min \sum \delta(i,j)}{\max \sum \delta(i,j) - \min \sum \delta(i,j)} \tag{8}$$

Ordering the values ξ_i provides the rank of each alternative. The best alternatives are those that have higher value ξ_i.
Next, we present results for a case study.

4. Experimental Results

4.1 Case study - Rental evaluation of residential properties

This case study considers the rental evaluation of residential properties in the city Volta Redonda, RJ, Brazil. The information of the decision matrix, which consists of crisp values, was adopted from Gomes & Rangel (2009). It is listed in Table 1 and is composed for 15 alternatives and 8 criteria. The weight vector is listed in Table 2. The original formulation of the TODIM presents limitations since it is not possible to take into account uncertainty into the decision matrix. Our formulation attempts to overcome these limitations by describing the uncertainty in form of trapezoidal fuzzy numbers. In our study we investigate the case of symmetric and asymmetric uncertainty.

Table 1. Decision matrix (Gomes and Rangel, 2009).

Alternatives	Criteria							
	C1	C2	C3	C4	C5	C6	C7	C8
A1	3	290	3	3	1	6	4	0
A2	4	180	2	2	1	4	2	0
A3	3	347	1	2	2	5	1	0
A4	3	124	2	3	2	5	4	0
A5	5	360	3	4	4	9	1	1
A6	2	89	2	3	1	5	1	0
A7	1	85	1	1	1	4	0	1
A8	5	80	2	3	1	6	0	1
A9	2	121	2	3	0	6	0	0
A10	2	120	1	3	1	5	1	0
A11	4	280	2	2	2	7	3	1
A12	1	90	1	1	1	5	2	0
A13	2	160	3	3	2	6	1	1
A14	3	320	3	3	2	8	2	1
A15	4	180	2	4	1	6	1	1

4.2 Decision matrix with symmetric uncertainty

To the original decision matrix listed in Table 1 was introduced -10%, -5%, +5%, +10% uncertainty to build up a_1, a_2, a_3, a_4, respectively in form of trapezoidal fuzzy number according to:

$a_1 = m - 0.1m,\ a_2 = m - 0.05m,\ a_3 = m + 0.05m,\ a_4 = m + 0.1m,$

where m stands for the mean graded (the original crisp value in the Table 1) of the trapezoidal fuzzy number. Fig. 3 depicts the case for the cell (1,1) corresponding to alternative A1 with respect to criterion C1 of the fuzzy decision matrix. The trapezoidal fuzzy decision matrix generated is listed in Table 3.

Table 2. Assigned weights to criteria (Gomes and Rangel, 2009).

C1	0.25
C2	0.15
C3	0.10
C4	0.20
C5	0.05
C6	0.10
C7	0.05
C8	0.10

Table 3. Fuzzy decision matrix with symmetric uncertainty.

Alternatives	Criteria			
	C1	C2	C3	C4
A1	[2.7, 2.85, 3.15, 3.3]	[261, 275.5, 304.5, 319]	[2.7, 2.85, 3.15, 3.3]	[2.7,2.85,3.15,3.3]
A2	[3.6, 3.8, 4.2, 4.4]	[162, 171, 189, 198]	[1.8, 1.9, 2.1, 2.2]	[1.8,1.9,2.1,2.2]
A3	[2.7, 2.85, 3.15, 3.3]	[312.3, 329.65, 364.35, 381.7]	[0.9, 0.95, 1.05, 1.1]	[1.8,1.9,2.1,2.2]
A4	[2.7,2.85, 3.15, 3.3]	[111.6, 117.8, 130.2, 136.4]	[1.8, 1.9, 2.1, 2.2]	[2.7,2.85,3.15,3.3]
A5	[4.5, 4.75, 5.25, 5.5]	[324, 342, 378, 396]	[2.7, 2.85, 3.15, 3.3]	[3.6,3.8,4.2,4.4]
A6	[1.8, 1.9, 2.1, 2.2]	[80.1, 84.55, 93.45, 97.9]	[1.8, 1.9, 2.1, 2.2]	[2.7,2.85,3.15,3.3]
A7	[0.9, 0.95, 1.05, 1.1]	[76.5, 80.75, 89.25, 93.5]	[0.9, 0.95, 1.05, 1.1]	[0.9,0.95,1.05,1.1]
A8	[4.5, 4.75, 5.25, 5.5]	[72, 76, 84, 88]	[1.8, 1.9, 2.1, 2.2]	[2.7,2.85,3.15,3.3]
A9	[1,8, 1.9, 2.1, 2.2]	[108.9, 114.95, 127.05, 133.1]	[1.8, 1.9, 2.1, 2.2]	[2.7,2.85,3.15,3.3]
A10	[1.8, 1.9, 2.1, 2.2]	[108, 114, 126, 132]	[0.9, 0.95, 1.05, 1.1]	[2.7,2.85,3.15,3.3]
A11	[3.6, 3.8, 4.2, 4.4]	[252, 266, 294, 308]	[1.8, 1.9, 2.1, 2.2]	[1.8,1.9,2.1,2.2]
A12	[0.9,0.95, 1.05, 1.1]	[81, 85.5, 94.5, 99]	[0.9, 0.95, 1.05, 1.1]	[0.9,0.95,1.05,1.1]
A13	[1.8, 1.9, 2.1, 2.2]	[144,152,168, 176]	[2.7, 2.85, 3.15, 3.3]	[2.7,2.85,3.15,3.3]
A14	[2.7, 2.85, 3.15, 3.3]	[288, 304, 336, 352]	[2.7, 2.85, 3.15, 3.3]	[2.7,2.85,3.15,3.3]
A15	[3.6, 3.8, 4.2, 4.4]	[162, 171, 189, 198]	[1.8, 1.9, 2.1, 2.2]	[3.6,3.8,4.2,4.4]

Table 3. Fuzzy decision matrix with symmetric uncertainty (cont.).

Alternatives	Criteria			
	C5	C6	C7	C8
A1	[0.9, 0.95, 1.05, 1.1]	[5.4, 5.7, 6.3, 6.7]	[3.6,3.8,4.2,4.4]	[0]
A2	[0.9, 0.95, 1.05, 1.1]	[3.6,3.8,4.2,4.4]	[1.8,1.9,2.1,2.2]	[0]
A3	[1.8, 1.9, 2.1, 2.2]	[4.5, 4.75, 5.25, 5.5]	[0.9,0.95,1.05,1.1]	[0]
A4	[1.8, 1.9, 2.1, 2.2]	[4.5, 4.75, 5.25, 5.5]	[3.6,3.8,4.2,4.4]	[0]
A5	[3.6, 3.8, 4.2, 4.4]	[8.1, 8.55, 9.45, 9.9]	[0.9,0.95,1.05,1.1]	[0.9, 0.95, 1.05, 1.1]
A6	[0.9, 0.95, 1.05, 1.1]	[4.5, 4.75, 5.25, 5.5]	[0.9,0.95,1.05,1.1]	[0]
A7	[0.9, 0.95, 1.05, 1.1]	[3.6, 3.8, 4.2, 4.4]	[0]	[0.9, 0.95, 1.05, 1.1]
A8	[0.9, 0.95, 1.05, 1.1]	[5.4, 5.7, 6.3, 6.7]	[0]	[0.9, 0.95, 1.05, 1.1]
A9	[0]	[5.4, 5.7, 6.3, 6.7]	[0]	[0]
A10	[0.9, 0.95, 1.05, 1.1]	[4.5, 4.75, 5.25, 5.5]	[0.9,0.95,1.05,1.1]	[0]
A11	[1.8, 1.9, 2.1, 2.2]	[6.3, 6.65, 7.35, 7.7]	[2.7,2.85,3.15,3.3]	[0.9, 0.95, 1.05, 1.1]
A12	[0.9, 0.95, 1.05, 1.1]	[4.5, 4.75, 5.25, 5.5]	[1.8,1.9,2.1,2.2]	[0]
A13	[1.8, 1.9, 2.1, 2.2]	[5.4, 5.7, 6.3, 6.7]	[0.9,0.95,1.05,1.1]	[0.9, 0.95, 1.05, 1.1]
A14	[1.8, 1.9, 2.1, 2.2]	[7.2, 7.6, 8.4, 8.8]	[1.8,1.9,2.1,2.2]	[0.9, 0.95, 1.05, 1.1]
A15	[0.9, 0.95, 1.05, 1.1]	[5.4, 5.7, 6.3, 6.7]	[0.9,0.95,1.05,1.1]	[0.9, 0.95, 1.05, 1.1]

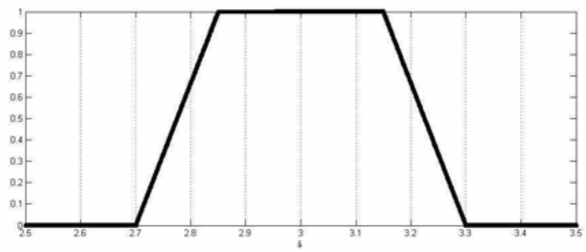

Fig. 3: Symmetric trapezoidal fuzzy number $\tilde{a} = (2.7,\ 2.85,\ 3.15,\ 3.3)$.

For comparisons purpose, we first apply the TODIM method to the original crisp matrix listed in Table 1 in order to get the reference ranking of the alternatives. Since the TODIM and the F-TODIM use the parameter θ, we adopt the reference value $\theta = 1$ (Gomes & Rangel, 2009). Next, we apply the F-TODIM method to the fuzzy trapezoidal matrix with symmetric uncertainty given in Table 3. In order to study the influence of the parameter θ, we also use the value $\theta = 2.5$ as suggested by Abdellaoui (2000). The ranking of the alternatives is shown in Table 4. In order to compare the results we adopt those results obtained from Gomes & Rangel (2009) as reference. We depict the ranking values for $\theta = 1$ and $\theta = 2.5$ in Fig. 4 and the prospect function in Fig. 5. As we can notice, the alternative A5 represents the best alternative for both methods. This means that the alternative A5 even though affected by uncertainty remains the best alternative. The results for $\theta = 1$ and $\theta = 2.5$ are almost the same, which indicates the robustness of the method. In general, the order of the alternatives obtained by F-TODIM compared to TODIM is different.

Table 4. Ranking of the alternatives.

Classification	TODIM crisp	Fuzzy TODIM a) symmetric $\theta = 1$	Fuzzy TODIM b) symmetric $\theta = 2.5$
1	A5	A5	A5
2	A14	A14	A14
3	A11	A11	A11
4	A13	A13	A13
5	A1	A15	A15
6	A15	A1	A1
7	A4	A4	A4
8	A8	A8	A8
9	A3	A2	A3
10	A2	A3	A2
11	A6	A6	A6
12	A10	A10	A10
13	A12	A9	A9
14	A9	A12	A12
15	A7	A7	A7

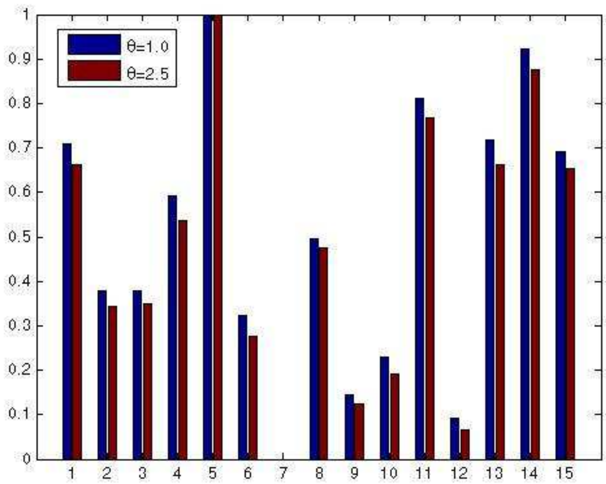

Fig. 4: Ranking of the alternatives for fuzzy trapezoidal matrix with 10% uncertainty around the mean.

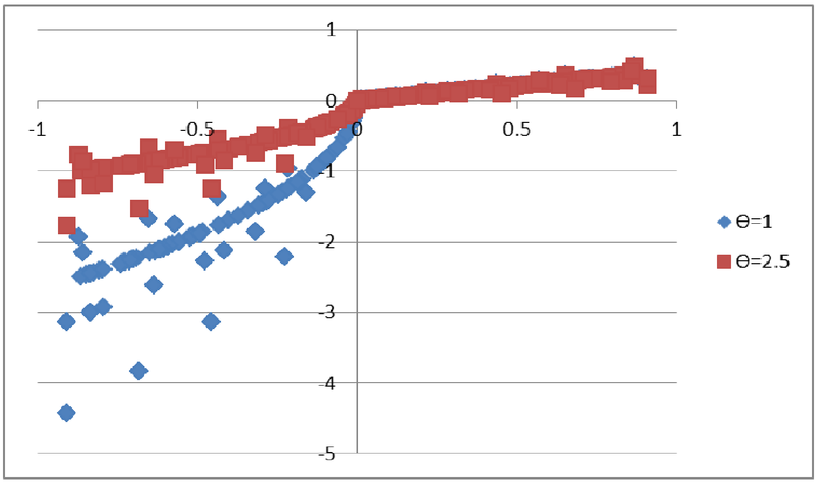

Fig. 5: Prospect function for trapezoidal fuzzy matrix with 10% symmetric uncertainty around the mean.

4.3 Decision matrix with asymmetric uncertainty

To the original decision matrix listed in Table 1 was introduced -8%, -3%, +2%, +7% uncertainty to build up $a_1, a_2, a_3, a_4,$ respectively in form of trapezoidal fuzzy number according to:

$a_1 = m - 0.08m,\ a_2 = m - 0.03m,\ a_3 = m + 0.02m,\ a_4 = m + 0.07m$

where m stands for the mean graded (the original crisp value in the Table 1) of the trapezoidal fuzzy number. Fig. 6 depicts the case for the cell (1,1) corresponding to alternative A1 with respect to criterion C1 of the fuzzy decision matrix. The trapezoidal fuzzy decision matrix generated is listed in Table 5.

Similar to the previous case, we apply the F-TODIM method to the fuzzy trapezoidal matrix with asymmetric uncertainty given in Table 5.The ranking is shown in Table 6. We depict the ranking values for $\theta = 1$ and $\theta = 2.5$ in Fig. 7 and the prospect function in Fig. 8.

Table 5. Fuzzy decision matrix with asymmetric uncertainty.

Alternatives	Criteria			
	C1	C2	C3	C4
A1	[2.76, 2.91, 3.06, 3.21]	[266.8, 281.3, 295.8, 310.3]	[2.76, 2.91, 3.06, 3.21]	[2.76, 2.91, 3.06, 3.21]
A2	[3.68, 3.88, 4.08, 4.28]	[165.6, 174.6, 183.6, 192.6]	[1.84, 1.94, 2.04, 2.14]	[1.84, 1.94, 2.04, 2.14]
A3	[2.76, 2.91, 3.06, 3.21]	[319.24, 336.59, 353.94, 371.29]	[0.92, 0.97, 1.02, 1.07]	[1.84, 1.94, 2.04, 2.14]
A4	[2.76, 2.91, 3.06, 3.21]	[114.08, 120.28, 126.48, 132.68]	[1.84, 1.94, 2.04, 2.14]	[2.76, 2.91, 3.06, 3.21]
A5	[4.6, 4.85, 5.1, 5.35]	[331.2, 349.2, 367.2, 385.2]	[2.76, 2.91, 3.06, 3.21]	[3.68, 3.88, 4.08, 4.28]
A6	[1.84, 1.94, 2.04, 2.14]	[81.88, 86.33, 90.78, 95.23]	[1.84, 1.94, 2.04, 2.14]	[2.76, 2.91, 3.06, 3.21]
A7	[0.92, 0.97, 1.02, 1.07]	[78.2, 82.45, 86.7, 90.95]	[0.92, 0.97, 1.02, 1.07]	[0.92, 0.97, 1.02, 1.07]
A8	[4.6, 4.85, 5.1, 5.35]	[73.6, 77.6, 81.6, 85.6]	[1.84, 1.94, 2.04, 2.14]	[2.76, 2.91, 3.06, 3.21]
A9	[1.84, 1.94, 2.04, 2.14]	[111.32, 117.37, 123.42, 129.47]	[1.84, 1.94, 2.04, 2.14]	[2.76, 2.91, 3.06, 3.21]
A10	[1.84, 1.94, 2.04, 2.14]	[110.4, 116.4, 122.4, 128.4]	[0.92, 0.97, 1.02, 1.07]	[2.76, 2.91, 3.06, 3.21]
A11	[3.68, 3.88, 4.08, 4.28]	[257.6, 271.6, 285.6, 299.6]	[1.84, 1.94, 2.04, 2.14]	[1.84, 1.94, 2.04, 2.14]
A12	[0.92, 0.97, 1.02, 1.07]	[82.8, 87.3, 91.8, 96.3]	[0.24, 0.32, 1.39, 1.74]	[0.92, 0.97, 1.02, 1.07]
A13	[1.84, 1.94, 2.04, 2.14]	[147.2, 155.2, 163.2, 171.2]	[0.92, 0.97, 1.02, 1.07]	[2.76, 2.91, 3.06, 3.21]
A14	[2.76, 2.91, 3.06, 3.21]	[294.4, 310.4, 326.4, 342.4]	[0.92, 0.97, 1.02, 1.07]	[2.76, 2.91, 3.06, 3.21]
A15	[3.68, 3.88, 4.08, 4.28]	[165.6, 174.6, 183.6, 192.6]	[1.84, 1.94, 2.04, 2.14]	[3.68, 3.88, 4.08, 4.28]

Table 5. Fuzzy decision matrix with asymmetric uncertainty (cont.).

Alternatives	Criteria			
	C5	C6	C7	C8
A1	[0.92, 0.97, 1.02, 1.37]	[5.52, 5.82, 6.12, 6.42]	[3.68, 3.88, 4.02 4.28]	[0]
A2	[0.92, 0.97, 1.02, 1.37]	[3.68, 3.88, 4.08, 4.28]	[1.84, 1.94, 2.04, 2.14]	[0]
A3	[1.84, 1.94, 2.04, 2.14]	[4.6, 4.85, 5.1, 5.35]	[0.92, 0.97, 1.02, 1.07]	[0]
A4	[1.84, 1.94, 2.04, 2.14]	[4.6, 4.85, 5.1, 5.35]	[3.68, 3.88, 4.02 4.28]	[0]
A5	[3.68, 3.88, 4.08, 4.28]	[8.28, 8.73, 9.18, 9.63]	[0.92, 0.97, 1.02, 1.07]	[0.92, 0.97, 1.02, 1.07]
A6	[0.92, 0.97, 1.02, 1.37]	[4.6, 4.85, 5.1, 5.35]	[0.92, 0.97, 1.02, 1.07]	[0]
A7	[0.92, 0.97, 1.02, 1.37]	[3.68, 3.88, 4.08, 4.28]	[0]	[0.92, 0.97, 1.02, 1.07]
A8	[0.92, 0.97, 1.02, 1.37]	[5.52, 5.82, 6.12, 6.42]	[0]	[0.92, 0.97, 1.02, 1.07]
A9	[0]	[5.52, 5.82, 6.12, 6.42]	[0]	[0]
A10	[0.92, 0.97, 1.02, 1.37]	[4.6, 4.85, 5.1, 5.35]	[0.92, 0.97, 1.02, 1.07]	[0]
A11	[1.84, 1.94, 2.04, 2.14]	[6.44, 6.79, 7.14, 7.49]	[2.76, 2.91, 3.06, 3.21]	[0.92, 0.97, 1.02, 1.07]
A12	[0.92, 0.97, 1.02, 1.37]	[4.6, 4.85, 5.1, 5.35]	[1.84, 1.94, 2.04, 2.14]	[0]
A13	[1.84, 1.94, 2.04, 2.14]	[5.52, 5.82, 6.12, 6.42]	[0.92, 0.97, 1.02, 1.07]	[0.92, 0.97, 1.02, 1.07]
A14	[1.84, 1.94, 2.04, 2.14]	[7.36, 7.76, 8.16, 8.56]	[1.84, 1.94, 2.04, 2.14]	[0.92, 0.97, 1.02, 1.07]
A15	[0.92, 0.97, 1.02, 1.37]	[5.52, 5.82, 6.12, 6.42]	[0.92, 0.97, 1.02, 1.07]	[0.92, 0.97, 1.02, 1.07]

As we can notice, the alternative A5 represents the best alternative for both methods. This means that the alternative A5 even though affected by uncertainty continues to be the better alternative. For $\theta = 1$ and $\theta = 2.5$ the order of the alternatives is almost the same. In general, the order of the alternatives obtained by F-TODIM compared to TODIM is different.

Table 6. Ranking of alternatives.

Classification	TODIM crisp	Fuzzy TODIM b) asymmetric $\theta = 1$	Fuzzy TODIM b) asymmetric $\theta = 2.5$
1	A5	A5	A5
2	A14	A14	A14
3	A11	A11	A11
4	A13	A13	A13
5	A1	A15	A15
6	A15	A1	A1
7	A4	A4	A4
8	A8	A8	A8
9	A3	A2	A3
10	A2	A3	A2
11	A6	A6	A6
12	A10	A10	A10
13	A12	A9	A9
14	A9	A12	A12
15	A7	A7	A7

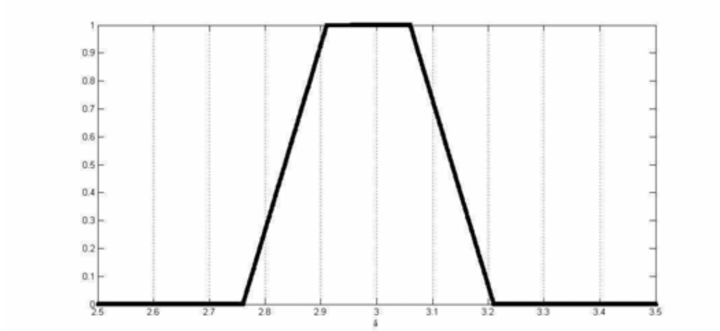

Fig. 6: Asymmetric trapezoidal fuzzy number $\tilde{a} = (2.76,\ 2.91,\ 3.06,\ 3.21)$.

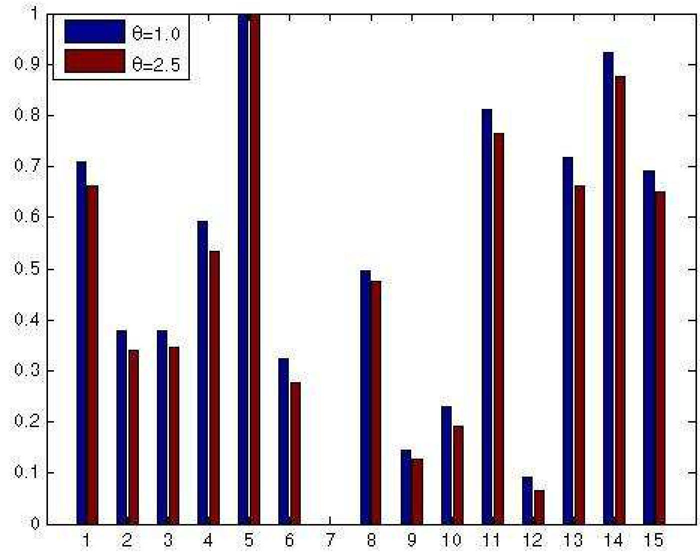

Fig. 7: Ranking of the alternatives for fuzzy trapezoidal matrix with asymmetric uncertainty.

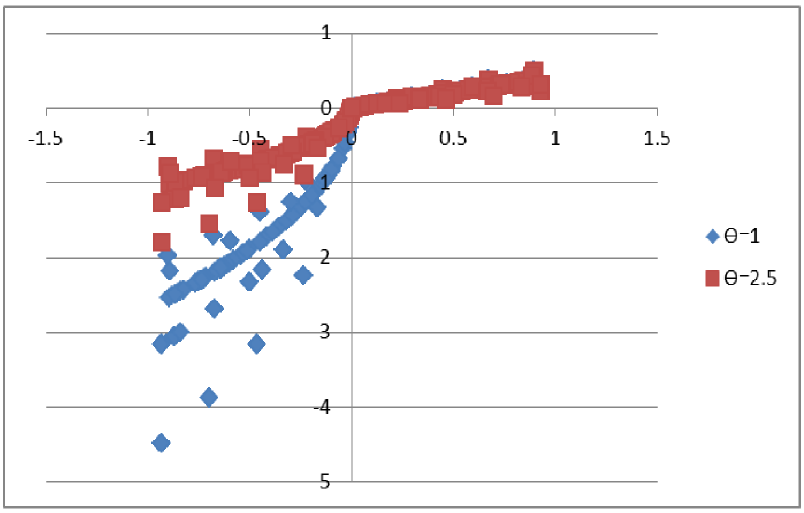

Fig. 8: Prospect function for fuzzy trapezoidal matrix with asymmetric uncertainty.

5. Conclusions

In this work, we have applied the fuzzy TODIM method, for short, F-TODIM for multi-criteria decision making to tackle problems affected by uncertainty. The F-TODIM has been investigated for a case study consisting of rental evaluation of residential properties where the decision matrix is represented by trapezoidal fuzzy numbers. In general, the order of the alternatives obtained by F-TODIM compared to TODIM is different since F-TODIM is a more general approach taking into account uncertainty. The standard TODIM, in its original formulation, is only applicable to crisp decision matrices. The F-TODIM method can be applied to more challenging MCDM problems considering uncertain environments. We currently are expanding the method to other applications.

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