APPLICATION OF KRIGING METHOD FOR ESTIMATING THE CONDITIONAL VALUE AT RISK IN ASSET PORTFOLIO RISK OPTIMIZATION

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ABSTRACT

This paper presents a new asset portfolio optimization model which considers the Conditional Value at Risk – CVAR as a risk measure. Based on Kriging Method, it creates a surface to describe CVAR. This new method led to results similar to those obtained by ROCKAFELLAR (2000). The results under stress scenarios were compared to Markowitz model.

Key-words: Operational Research, Optimization, Portfolio Management, Finance

Área principal: GF - Financial Management, MP - Probabilistic Models, SIM - Simulation

1. INTRODUCTION

This paper introduces a new model for calculating the optimal investment portfolio composition. The new model, based on the Kriging Model, is believed to be better approach to the portfolio composition problem since it does not assume a predetermined response probability distribution. Besides, it requires substantially less variables and constraints to reach similar results to well known models. Moreover, this new model calculates the relative error to the future estimates, offering a substantial contribution to the models available in the literature.

Portfolio Management Theory, introduced by MARKOWITZ (1952), considers the variance of portfolio returns as a risk measure. Though this model has been used by practitioners for decades, it's well known that this risk measure applies only to symmetrical distributions of returns (SZEGO, 2002), which does not apply to most of the real cases. The *CVAR Model*, introduced by ROCKAFELLAR and URYASEV (2000), considers the Conditional Value at Risk (CVAR) as a risk measure, overcoming the limitations of the previous model. Moreover, it proposes a mathematical formulation that transforms the problem of calculating the CVAR in a linear programming problem. However, the discretization of the problem requires a large number of variables and restrictions, which grows with the sample size. The proposed model also utilizes the CVAR as a risk measure, but instead of considering an analytical expression for CVAR in terms of probability distribution of returns, it considers estimates of CVAR calculated in a grid of R^N, representing the different portfolios. These estimates are used to construct a response surface - the risk surface - which is the objective function of the optimization model.

The three models mentioned above are applied to a portfolio composed of four stocks negotiated in BM&F BOVESPA. The results of such application show that the primary investment recommendations obtained by the three presented models are the same. Moreover, the calculated portfolio compositions are similar, and the portfolios calculated by the Kriging Model rely somewhere in between the previous models. Therefore, the new proposed model can be considered more appropriate to the portfolio composition problem since it presents a simpler solution than the CVAR Model in terms of variables and restrictions, and it calculates the estimate error, reaching similar results.

2. PORTFOLIO COMPOSITION MODELS

Portfolio Management Theory introduced by MARKOWITZ (1952) studies the problem of composing an optimal portfolio of financial assets. The objective is to find a portfolio with minimum risk, satisfying constraints on the return of the portfolio. Markowitz considers the variance of portfolio return as the risk measure. Although this model has been used for decades, it is well known that this measure applies only to symmetrical distributions of returns (SZEGO (2002)) and does not consider the tail of distributions.

Some other risk measures have been considered in literature, such as the **Value at Risk - VAR**. The VAR can be defined as greatest loss that might occur with α% of probability in a fixed time horizon, but it does not analyze the tail of the resulting return distribution. ROCKAFELLAR (2000) proposed a new risk measure to cover the limitations presented, the **Conditional Value at Risk – CVAR**. CVAR is a coherent risk measure (ARTZNER (1999)) which can be defined as the average of values that exceed the VAR, for a fixed confidence level.

The portfolio problem can be generally written as:

Minimize: $risk(X)$

Subject to:
(1) $\sum_{i=1}^{n} E(R_i)x_i \geq G$
(2) $\sum_{i=1}^{n} x_i = 1$
(3) $x_i \geq 0, i = 1, \ldots, n$

Where x_i is the weight of asset i in the portfolio, $E(R_i)$ is the expected return of the asset i, where $i = 1,2, \ldots, n$, and $risk(X)$ is a risk measure written in terms of portfolio composition.

Constraint (1) refers to the average return of the portfolio, which must be equal or greater than a fixed return level. Constraint (2) assures that the total amount of available resources will be invested. Constraint (3) does not allow short sales.

Considering the variance as risk measure we can write $risk(X) = x^t \sum x$, ($\sum$ is the covariance of returns) and the model can be solved by Quadratic Optimization techniques. If we consider the VAR as risk measure, the function $risk(X)$ presents local minimum and the resulting model is difficult to solve (RIBEIRO (2004)).

ROCKAFELLAR and URYASEV (2000) considered Conditional Value at Risk – CVAR as risk measure. According to QUARANTA (2008), considering $x \in X \subset R^N$ a decision vector representing a portfolio, $y \in Y \subset R^N$ the future return of the assets that form a portfolio and $z = f(x, y)$ the function of portfolio losses, the CVAR is calculated as:

$$CVAR(x, a) = E\{y | f(x, y) \leq a\}$$

Where a is the VAR of the portfolio. ROCKAFELLAR (2002) showed that CVAR can be written as $F_\alpha(X, a) = CVAR(X, a) = a + \frac{1}{1-\alpha} \int_{Y \in R^m} [f(X, Y) - a]^+ p(Y) dy$, with

$[t]^+ = max\{0, t\}$ and α the confidence level.

ROCKAFELLAR (2002) proposed a mathematical formulation that transforms the problem of calculating the CVAR in a linear programming problem. However this approach assumes discretization of the probability distribution Y, generating vectors $y_1, y_2, \ldots, y_q$. Given the q scenarios, the approximated function for $F_\alpha(X, a)$ becomes:

$$\hat{F}_\alpha(X,a) = CVAR(X,a) = a + \frac{1}{q(1-\alpha)}\sum_{k=1}^{q}[f(X,Y_k) - a]^+$$

It is easy to see that the resulting optimization model can be transformed in a linear programming model with the introduction of auxiliary variables μ_k.

Minimize: $\hat{F}(X,a) = a + \frac{1}{q(1-\alpha)}\sum_{k=1}^{q}\mu_k$

Subject to:
(1) $x_i \geq 0 \; for \; j = 1, \dots, n$
(2) $\sum_{j=1}^{n} x_j = 1$
(3) $X^T Y \geq G$
(4) $\mu_k + X^T Y_k + a \geq 0$
(5) $\mu_k \geq 0, k \in \{1,2,\dots,q\}$

Despite the advantages CVAR model over Markowitz model, it offers one major disadvantage: the need to include a large number of variables and restrictions, which grows with the sample size.

3. A NEW PORTFOLIO MODEL

Instead of considering an analytical expression for CVAR in terms of probability distribution of returns, this paper proposes a new approach. We consider estimates of CVAR calculated in a grid of R^N, representing different portfolios. These estimates are used to construct a response surface - the risk surface - which is the objective function of the optimization model. The response surface is created based on Kriging Model (RIBEIRO (2004), YIN (2011)).

Given q pairs $\{(w^{(j)}; y^{(j)})\}_{j=1}^{q}$, with $w^{(j)} \in R^n$ and $y^{(j)} = f(w^{(j)})$, the Kriging Model creates a polynomial approximation of a function f(.), $\sum_{j=1}^{q}\alpha_j f^{(j)}(w) + e(w)$. The random errors $e(w)$ are correlated, normally distributed, with mean equal to zero and variance σ^2. The covariance between the errors is given by $cov\left(e(X^i), e(X^j)\right) = \sigma^2 \sum_{ij}$, where $\sum_{ij}$ is the correlation between two errors ($\sum_{ij} = R(\theta, d_h) = corr(X^j, X^j)$).

An unbiased estimator of $f(X^*)$ is given by RIBEIRO (2002) and LOPHAVEN (2002):
$\hat{f}(X^*) = \sum_{j=1}^{m}\beta_j^* f^j(X^*) + r'\Sigma^{-1}(y - F\beta^*)$ with $\beta = (F^T\Sigma^{-1}F)^{-1} F^T\Sigma^{-1}y$. Vector r is the correlation vector between errors related to X^* and the other points in the sample, $\sum$ is the correlation matrix between the points in the sample, y is the vector of the values observed for CVAR, and F is the matrix with the values calculated in the points of the sample.

The Gaussian correlation is given by $R(\theta, d_h) = \exp(-\theta_h d_h^2)$ with the distance measure between two points depending on parameters θ_h and p_h:

$$d(X_i, X_j) = \sum_{h=1}^{n}\theta_h |x_h^i - x_h^j|^{p_h}$$

According to JONES (1998), the θ_h parameter measures the influence or "activity" of the variable x_h. The exponent p_h is related to the smoothness of the function in relation to the points h. Similarly to QUEIPO (2002), we adopt $\theta_h = 1$ and $p_h = 2$.

The first step to the application of the model relates to the definition of an appropriate sample to the experiment. Two techniques are considered:

1. Random Generation: generates points that are normally distributed in the interval [0,1], the mean equals to zero and the variance equals to one.
2. Deterministic Generation: each face of the hypercube $[0,1]^n$ is subdivided in a defined number of intervals which generate other cubes which vertices are the points of the sample (RIBEIRO (2004)).

Once the sample generation technique is chosen, the correlation and the regression are defined and the model is constructed as follows:

a. Generate a sample of points $\{X^i\}_{i=1}^{q}$, satisfying the constraints $X^i \in R^n$ and $0 \leq x_j^i \leq 1, i = 1, \dots, n;$
b. For each vector X^i, calculate $y^i = CVAR(X^i)$
c. Construct the approximating function $(X^i; Y^i)$;
d. Solve the resulting optimization model

4. RESULTS

To analyze the model, a portfolio with four stocks negotiated in BM&F BOVESPA was considered**:** Cosan LTD (CZLT11), Itaú Unibanco PN (ITUB4), BMFBovespa ON (BVMF3) and Gerdau PN (GGBR4).

The sample considers daily returns of each stock from October 2009 to August 2011, totalizing 474 points. A hypothesis test showed that returns were not normally distributed at 5% significance level. During the considered period, three trend scenarios were clearly defined: the first scenario was of relative price stability, the second of price rise, and the third of price decrease of the stocks. The figure above shows the three considered scenarios.

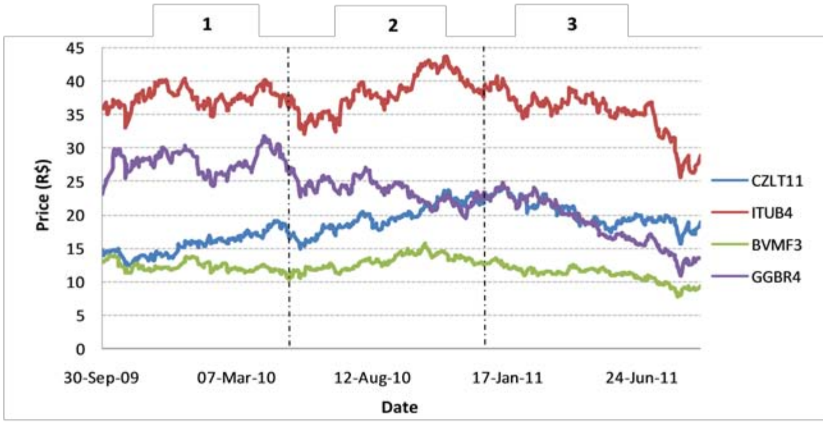

Figure 1: Definition of three scenarios for analysis.

4.1 The Markowitz Model

To apply the Markowitz Model two parameters were considered for each scenario: (1) the mean return of each stock for each scenario, and (2) the covariance matrix between the four assets for each scenario. The covariance may be considered constant in the three different scenarios.
Applying the formulation proposed by MARKOWITZ (1952) to each of the three scenarios, the obtained Efficient Frontiers are presented below.

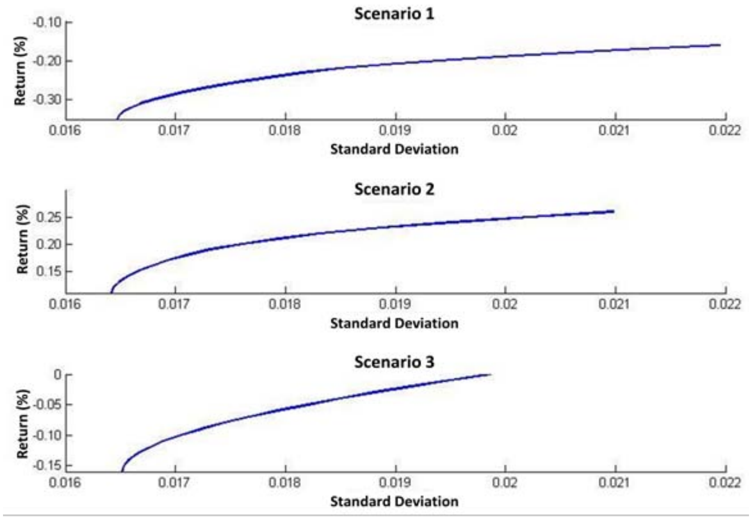

Figure 2: Efficient Frontiers for each scenario using the Markowitz Model.

The optimal solutions are represented as follows:

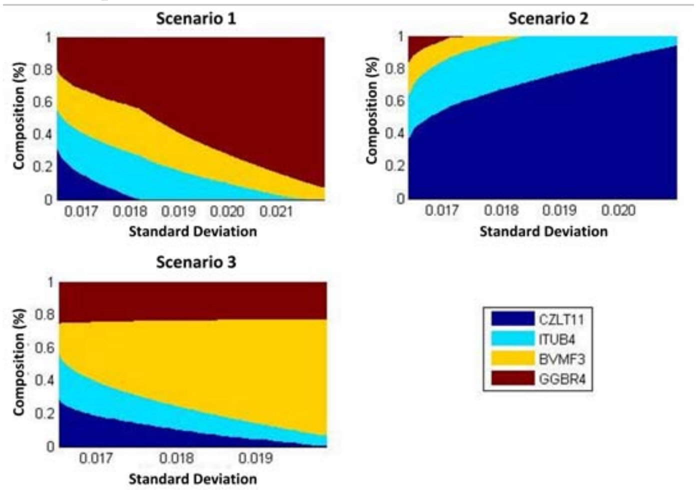

Figure 3: Optimal Portfolio Composition for each scenario calculated using the Markowitz Model.

The above results were obtained with the solution of a quadratic optimization problem with 4 variables and 3 restrictions.

4.2 The CVAR Model

To apply the CVAR Model we assumed a confidence level of 90% and a fixed time horizon of 1 day. Similarly to the previous section, the efficiency frontiers of the optimal portfolio, for each scenario, are presented below.

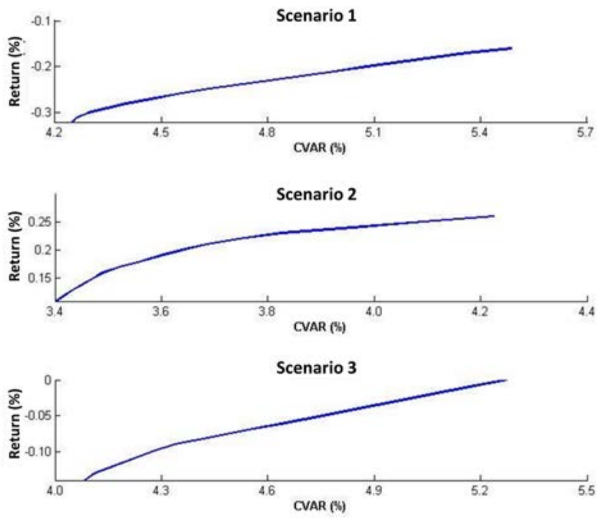

Figure 4: Efficient Frontiers for each scenario using the CVAR Model.

Below the portfolio compositions calculated using the CVAR method are presented.

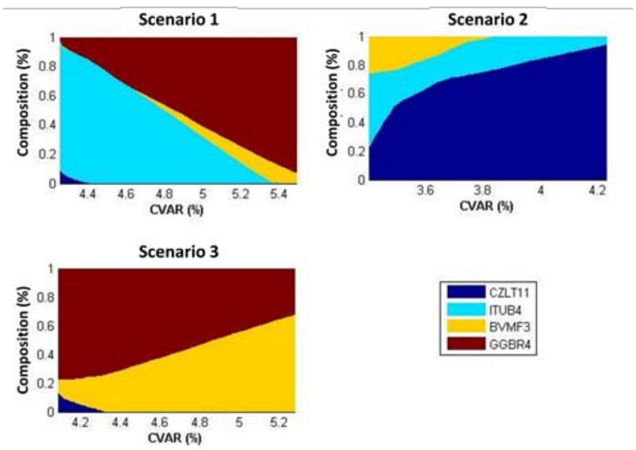

Figure 5: Optimal Portfolio Composition for each scenario calculated using the CVAR Model.

The above results were obtained with the solution of a discrete linear optimization problem with 100 variables and 104 restrictions (100 simulations performed).

4.3 Kriging Model

4.3.1 Estimating the CVAR

The first step to apply the Kriging model consists in obtaining the CVAR values of the portfolio formed by the combination of assets previously defined. VAR was obtained through historical data (non-parametric method) with 90% confidence level. For the historical returns, the average daily return between October 2009 and August 2011, totalizing 474 observations, was used. The CVAR is calculated as the average of the returns inferior to those defined for the VAR.

4.3.2 Sample Selection

Two sample generation methods were considered to apply Kriging method. In the *Random Method*, the portfolio compositions for the four defined assets were generated randomly, so that $0 \le x_j^{(i)} \le 1$, forming a sample of 100 different portfolio compositions. In the *Deterministic Method*, the values of $x_j^{(i)}$ were defined so that they were equally distributed in the analyzed region. The condition $\sum_{j=1}^{4} x_j^{(i)}$ was relaxed to **facilitate** graphical representation.

The figure below represents the samples referring to the stocks ITUB4 and CZLT11. These assets were chosen arbitrarily to make the representation easier.

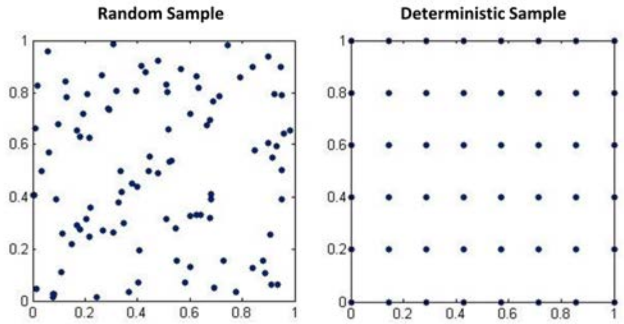

Figure 6: Graphic representation of the Random Sample and Deterministic Sample for two arbitrarily define assets.

The CVAR for the portfolios was determined using both samples. It was calculated as the average of returns lower than the VAR, which was defined by percentile at a confidence level of 90% and the fixed horizon of 1 day. Next, an approximation of the Conditional Values at Risk was calculated, according to the Kriging model with the four assets. The figure below shows the response surfaces generated for both samples. We selected only the points in which allocations in BVMF3 and GGBR4 were 25% and 50%, to facilitate illustration.

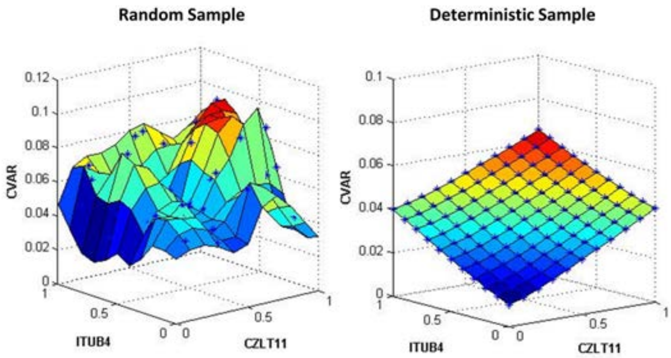

Figure 7: Smoothened response surfaces when applied the DACE method.

The differences in the sample methods that represent the portfolio composition of each asset may be observed by the error calculated as:

$$\hat{f}(X) - CVAR(X)$$

Where $\hat{f}(X)$ represents the value of the CVAR approximated by the DACE adjustment. The figure below shows this error.

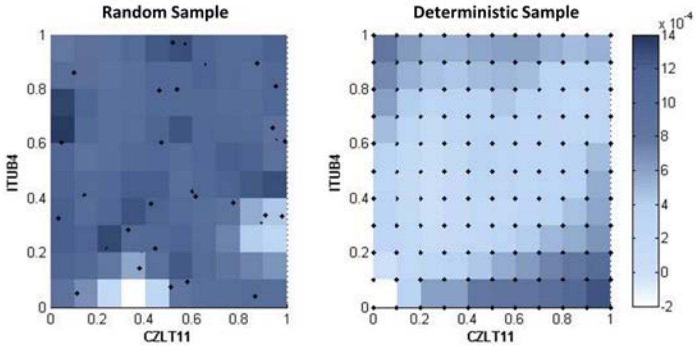

Figure 8: Error of the Kriging approximation for random and deterministic samples.

Figure 8 shows the high sensibility of the Kriging method to the sample definition methodology. As the random sample offers greater error than the deterministic sample, CVAR values calculated using the deterministic sample will be used in the optimization models.

4.3.3 Correlation Function Selection

All correlation functions proposed by LOPHAVEN (2002) were analyzed and Gaussian Correlation function was selected as it led to the smallest error.

4.3.4 The Kriging Model Application

The Kriging Model, which interpolates the data, depends on the sample points, the type of regression, the correlation model and the value of θ. These parameters are presented below.

- The sample is composed by the combination of four assets previously chosen, and was generated by deterministic process, which divides the surface grid in n equidistant intervals, defining the distance between two consecutive points $x_j^{(i)}$;
- The CVAR was calculated as the average of the values inferior to the calculated VAR, which was defined by the historical series method;
- The defined regression method was the linear regression, which utilizes a polynomial of degree 1 to approximate the function;
- The defined correlation method was the Gaussian, since this offers a better approximation of the function according to the value of the mean squared error;
- The definition of this correlation method implies in the determination of the p_h, which is equal to 2 in the case of the Gaussian correlation;
- In an approach similar to QUEIPO (2002), it is defined $\theta = 2$.

The figure below shows the approximated surface for all three analyzed scenarios.

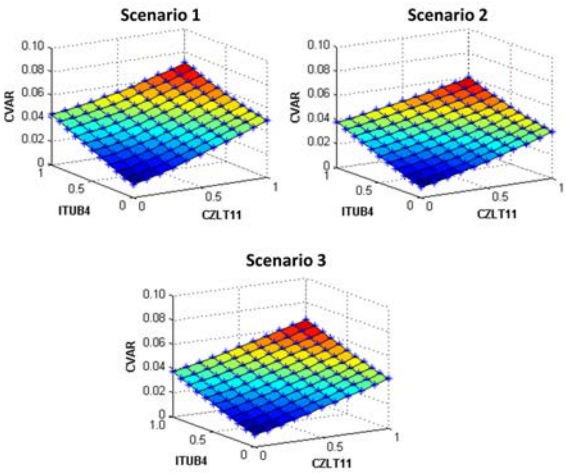

Figure 9: Approximation of the Kriging Method to the function of CVAR.

4.3.5 The Efficient Frontier Estimate

The figure below shows the results of the study.

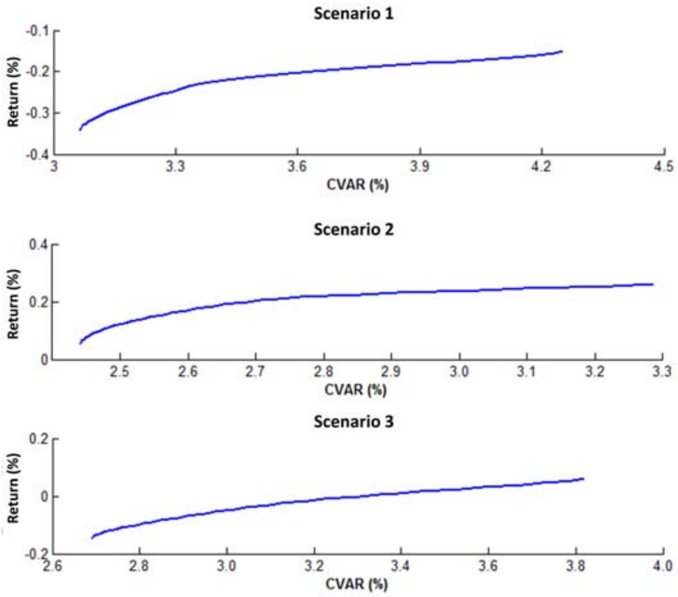

Figure 10: Calculated Efficient Frontiers for each scenario using the Kriging Model.

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The figure below shows the different portfolio compositions for different risk levels calculated using the Kriging Model.

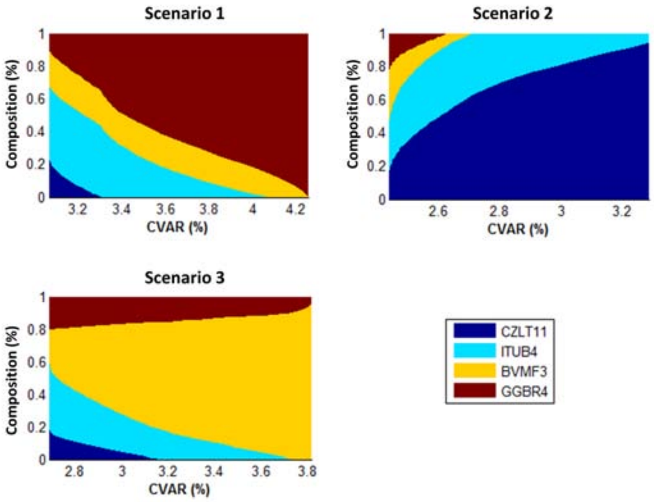

Figure 11: Optimal Portfolio Composition for each scenario using the Kriging Model.

The above results were obtained with the solution of a linear optimization problem with 4 variables and 3 restrictions.

The efficiency frontier recommends greater investment in GGBR4, CZLT11 and BVMF3 for scenarios 1, 2, and 3 respectively, as the investor's risk aversion reduces.

Comparing the portfolio compositions of the three analyzed methods, it's clear that the results are similar in terms of general investment allocations within the portfolio. Specifically, when comparing the Markowitz Model with the Kriging Method, the greatest squared deviation of portfolio composition for a given level of return is 0,042, demonstrating a strong similarity in the results offered by these two methods. The similarity of the results between the Markowitz Model and the Kriging Method is a strong indication of the soundness of the latter method.

When comparing the results of both the Markowitz Model and the Kriging Method with the CVAR Model, the greatest squared deviation of portfolio compositions is 0,064 for any given level of return in the right-hand half of the curve, also indicating a strong similarity between the three methods when considering high risk levels. However, in the left-hand half of the curves, the squared deviation between its results and the results of the other two models is as great as 0,38. Because this specific region is of great concentration of points, the difference might be explained by the relatively small number of simulations utilized to apply the CVAR model (around 100). An increase on the number of simulations of the CVAR model should make the results even more similar, but with the cost of further increasing in the number of variables and restrictions, and thus computational capacity.

The above discussion reinforces the advantages of the Kriging Model, which can generate sound risk analysis with the need of substantially less variables and restrictions, less computational capacity and without the need of estimating a determined probability distribution which doesn't necessarily occur.

7. CONCLUSIONS

This study introduced the Kriging Model as an alternative to the investment portfolio optimization models existing in today's literature. Though the Kriging Model uses the CVAR as a risk measure, similarly to the CVAR model, it requires substantially less variables and constraints in the optimization problem than the latter, representing a more appropriate solution. Moreover, it calculates the error of future estimates, offering a substantial contribution to the models available in the literature.

Together with the Markowitz Model and the CVAR Model, the Kriging Model was applied to a portfolio composed of four assets negotiated in BM&F BOVESPA, in three different scenarios: (1) price stability, (2) price rise and (3) price decline (stress scenario). The results show similar primary recommendations by each model: greater investment in GGBR4, CZLT11 and BVMF3 for scenarios 1, 2, and 3 respectively. This indicates that the portfolio compositions obtained by the new proposed model are similar to those obtained by the Markowitz Model and the CVAR Model, especially when considering high risk levels. In fact, the portfolio compositions calculated by new model rely somewhere in between the portfolio compositions calculated using the Markowitz Model and the CVAR Model, except for the price stress scenario, in which the new model's results are similar to those calculated using the Markowitz Model. This results were obtained with considerably less variables and restrictions than the CVAR Model, facilitating it's application. Last, the Kriging Model considers the relative error to the estimate, offering a substantial opportunity do further developments and extensions.

Other extensions of this work include the application of the Kriging Model without the hypothesis of homogeneous variance of the errors, refining the method and the results. One last extension of this work would be the application of the Kriging Model to multiple future return periods, which is not presented in this work.

References

ARTZNER, P., DALBAEN, F., EBER, J.-M., & HEATH, D. (1999). Coherent mesures of risk. *Mathematical Finance , 9*, 203-228.

JONES, D., SCHONLAU, M., & WELCH, W. (1998). Efficient global optimization of expensive black-box functions. *Journal of Global Optimization , v.13, n.4*, pp. p.455-492.

LOPHAVEN, S., NIELSEN, H., & SONDERGAARD, J. (2002). *DACE: a matlab kriging tollbox.* Denmark: Informatics and Mathematical Modelling, Technical University of Denmark.

LUENBERGER, D. G. (1998). *Investment Science.* New York: Oxford University Press.

MARKOWITZ, H. (1952). Portfolio Selection. *The Journal of Finance , v.7*, pp. p.77-91.

PFLUG, G. C. (2000). Some remarks on the Value-at-Risk and Conditional Value-at-Risk. In S. URYASEV, *Probabilistic constrained optimization* (pp. 272-281).

QUARANTA, A. G., & ZAFFARONI, A. (2008). Robust optimization of condition value at risk and portfolio selection. *Journal of Banking and Finance , v.32*, pp. 2046-2056.

QUEIPO, v. N., PINTOS, S., RÍNCON, N., CONTRERAS, N., & COLMENARES, J. (2002). Surrogate Modeling-based optimization for the integration of statistic and dynamic data into a reservoir description. *Journal of Petroleum Science and Engineering , v.37, n.3-4*, pp. p.167-181.

RIBEIRO, C. O., & FERREIRA, L. A. (2004). Uma contribuição ao problema de composição de carteiras de mínimo valor em risco. *Gestão e Produção , v.12*, pp. p.295-304.

ROCKAFELLAR, R. T., & URYASEV, S. (2002). Conditional Value-at-Risk for general loss distributions. *Journal of Banking and Finance , v.26*, pp. 1443-1471.

ROCKAFELLAR, R. T., & URYASEV, S. (2000). Optimization of Conditional Value-at-Risk. *Journal of Risk , v. 2*, pp. 21-41.

SZEGÖ, G. (2002). Measures of risk. *Journal of Banking & Finance , 26*, 1253-1272.

YIN, J., Ng, S., & Ng, K. (2011). Kriging metamodel with modified nugget-effect: The heteroscedastic variance case. *Computers & Industrial Engineering , 61*, pp. 760–777.