

A Multistage Linear Stochastic Programming Model for Optimal Corporate Debt Management

Abstract

Large corporations fund their capital and operational expenses by issuing bonds with a variety of indexations, denominations, maturities and amortization schedules. We propose a multistage linear stochastic programming model that optimizes the bond issuance policy to fund a predetermined project portfolio. Cash flows are uncertain, affected by financial, macroeconomic and business specific risk factors. Our objective function combines a mean-risk trade-off measured at the end of the planning horizon and penalties for highly leveraged debt portfolios at each intermediate stage. Avoiding the curse of dimensionality common in long term multistage stochastic models, the full event tree representation of uncertainty is used only in the initial planning stages. The remainder of the horizon uses independent path sub-samples, adopting a fixed-rule policy approximation. Assuming null cash returns, an illustrative example presents a sensitivity analysis of the first stage solution and the stochastic efficient frontier of mean-risk trade-off. Based on a realistic example with stochastic cash returns, we underscore the importance of the intermediate penalties in obtaining suitable solutions. Based on the proposed model, a financial planning software tool has been implemented and deployed in Brazilian oil company Petrobras.

Keywords: Risk management, Finance, Corporate bond issuance, Debt Management, Stochastic programming

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1. Introduction

In large corporations, the goal of debt management is the dynamic bond issuance under uncertainty, with the purpose of optimally funding their capital and operational expenses. Debt portfolios are structured as a mix of securities with differing indexations, denominations, maturities and amortization schedules, in an attempt to balance the expected cost of servicing the debt with risks inherent to interest rates, corporate revenues and costs. In addition to corporate and regulatory operational constraints, debt management must take into account fluctuations in total debt, assets and cash savings, along with other financial performance measures affecting the company's stock price and credit rating. In face of the required modeling flexibility, there is a firmly established literature with applications of *Multistage Stochastic Programming* (MSP) techniques to debt management and, more generally, to *Asset Liability Management* (ALM) problems. Starting with Bradley and Crane [2], ALM models have been developed for several different applications including insurance companies [3] and pension funds [11, 6, 10]. More recently, similar techniques were specialized for optimal sovereign bond issuance, also dealing with the trade-off between minimum expected cost and minimum risk [1, 4, 5]. For the corporate case, Xu and Birge [22] introduce a simplified model that maximizes shareholder value over production strategy and dividend distribution policy, considering a single short term debt instrument. However, their model requires the availability of known risk neutral probabilities, an unrealistic assumption especially for companies without a portfolio of tradable assets. To the best of our knowledge, the literature lacks models describing corporate bond issuance under uncertainty, dealing with both the complexity of the dynamic decision process and the trade-off among expected costs, risks and financial performance measures, as observed in practice.

In this article, we present an MSP model for a corporation financing a predetermined set of projects, considering a universe of fixed and floating rates debt instruments. Uncertainty is represented by an event tree with a hybrid information structure, used to avoid exponential complexity with the number of stages. In the first part of the horizon, we build a detailed event tree with a full range of debt instruments available to the decision maker. For the other portion of the time horizon, the event tree is formed by a subsample approximation of uncertainty realizations, with a predetermined policy rule allowing only short-term debt. Our optimization model describes the dynamic decision process where, at every yearly stage, the state of the system is represented by the current cash holdings and the past debt portfolio. It takes into account the mean-risk trade-off between expected cost of debt service and expected value of corporation insolvency. Additional operational constraints express corporate debt valuation and the current asset value used to compute the leverage ratio at each stage. Lewellen and Emery [12] asserts that most reasonable characterizations of corporate debt management policies adopt a borrowing strategy organized around leverage ratio targets. We integrate this performance measure into the objective function, modeling it as a convex piecewise linear penalty of the computed excess leverage.

For an illustrative example with null cash returns and no intermediate penalties, we present a sensitivity analysis of the risk aversion level. Considering different scenario trees, we solve the problem for each risk aversion level and compute efficient frontiers

and related solutions. For a realistic example with stochastic cash returns, we make a sensitivity analysis of the excess leverage penalties and show the importance of our multi-criteria objective function to obtain suitable policies. Computations were carried out with a financial planning software tool implemented for a financial and risk management group at Brazilian oil company Petrobras. In our illustration we consider a fictitious, although realistic, project data set.

The remaining content of this article is organized as follows. Section 2 describes our multistage stochastic programming model, with a comprehensive presentation of all elements in the formulation. In Section 3, we perform a series of sensitivity analyses of the optimal solution considering an illustrative example. Section 4 we present the assumptions of a realistic application of our model to the oil industry and show the importance of the excess leverage penalties. Finally, section 5 summarizes the contributions of this paper and outlines the directions of future research.

2. Multistage Stochastic Programming Model

Multistage stochastic programming is a natural framework for long-term financial planning problems, corporate debt management in particular. The model must describe a dynamic setting where, at a given stage, a decision is taken facing an unknown future. Once decisions are implemented, the next period information is revealed and the process is repeated for the next stage. Figure 1 illustrates this dynamic decision process where uncertainty gradually reveals itself over time.

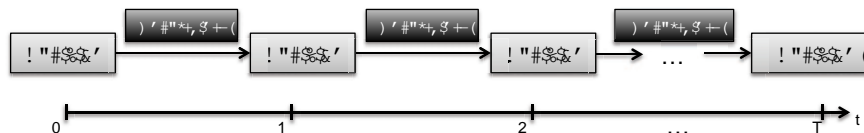


Figure 1: Dynamic decision process

A standard approach in MSP models is to represent uncertainty by a discrete event tree as depicted in Figure 2, where nodes indicate the state of the process at decision points and arcs the realizations of uncertainty before the next stage. Formally, the information structure given by an event tree can be understood as a filtered probability space [4] generating a deterministic equivalent of the MSP model. A complete path in the event tree is called a *scenario* and a policy is defined as the set of decisions for all stages and scenarios. This information structure requires that decisions be based solely on past information, expressed in the MSP model formulation by the *non-anticipativity* constraints, which stipulate decision variables at a given stage must be equal if their scenarios share the same node in the event tree. For instance, given a generic policy $X_t(s), \forall t \in \{0, 1, 2\}, s \in \{1, 2, 3, 4\}$ and the information structure in Figure 2, we would include $X_0(1) = X_0(2) = X_0(3) = X_0(4), X_1(1) = X_1(2)$ and $X_1(3) = X_1(4)$ as non-anticipativity constraints.

Given this tree structure, we can immediately observe that the size of the deterministic equivalent grows exponentially with the number of stages. Some authors have

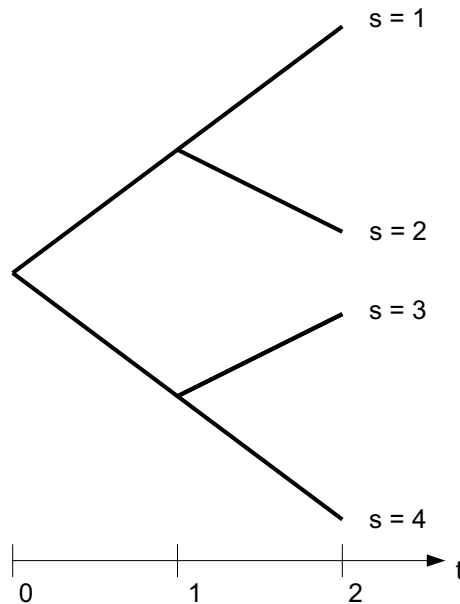


Figure 2: Information structure given by an event tree

dealt with this *curse of dimensionality* applying large scale optimization techniques [15, 16], while others approximate the original multistage problem by reducing the number decision variables with the adopting single policy rule [19]. A policy rule is a function of the uncertainty realization that generates a unique sequence of feasible decisions for each time of the planning horizon. This framework fits into the independent scenario structure as stated in [19], however it usually leads to a suboptimal solution when compared to the original multistage one. Indeed, one could define a set of policy rules generally leading to a non-convex optimization problem.

With the purpose of reducing the high dimensionality of our final formulation, we propose a hybrid approach comprising a traditional multistage model for the first T^* periods and an independent-scenario structure with simple policy rule for $t > T^*$. For the latter, we represent uncertainty by a subsample of the full event tree structured as independent scenarios as illustrated in Figure 3. In our model, a full set of securities is considered for $t \leq T^*$, while for $t > T^*$ we allow only short term bonds to ensure the minimum cash threshold. This framework is motivated by the assumption that most investments take place at the first part of the planning horizon where the decision process is described in more detail.

2.1. Definitions

Preparing a complete formal statement of the model, let us first define parameters, risk factors and decision variables used in the formulation.

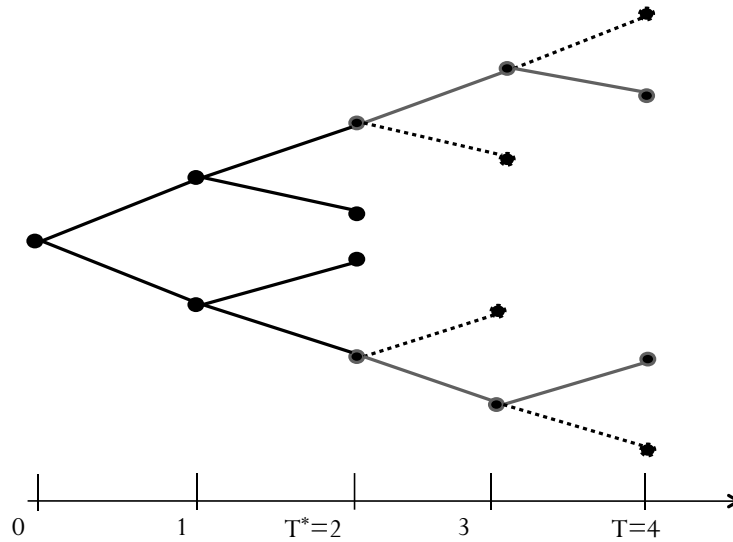


Figure 3: Hybrid model for $T^* = 2$ and $T = 4$

Scalar Parameters.

T : Planning horizon

T^* : Detailed planning horizon for $t = 0, \dots, T^* - 1$

S : Number of scenarios

ω : Weighted average cost of capital (WACC)

c : Initial cash

p : Risk aversion parameter (penalty coefficient for insolvent scenarios)

nX : Number of fixed rate bonds

nY : Number of floating rate bonds

K : Number of targets for the leverage ratio

Sets.

$\mathcal{H} = \{0, \dots, T - 1\}$

$\mathcal{H}^* = \{0, \dots, T^* - 1\}$

$\mathcal{S} = \{1, \dots, S\}$

$\mathcal{K} = \{1, \dots, K\}$

$$\mathcal{X} = \{1, \dots, nX\}$$

$$\mathcal{Y} = \{1, \dots, nY\}$$

Vector parameters.

γ_k : k -th target for leverage ratio, $\forall k \in \mathcal{K}$

θ_k : Penalty for excess leverage exceeding the k -th target, $\forall k \in \mathcal{K}$

x_t : Payment at $t \in \mathcal{H} \cup \{T\}$ of pre-existing fixed-rate bonds

y_t : Outstanding face value at $t \in \mathcal{H} \cup \{T\}$ of pre-existing floating rate bonds

Δy_t : Amortization at $t \in \mathcal{H} \cup \{T\}$ of the pre-existing floating rate bonds

M_X^i : Maturity of fixed rate bond i , $\forall i \in \mathcal{X}$, where $M_X^i \leq T - T^* + 1$

M_Y^i : Maturity of floating rate bond i , $\forall i \in \mathcal{Y}$, where $M_Y^i \leq T - T^* + 1$

ΔX_j^i : Amortization rate of fixed rate bond i , $\forall i \in \mathcal{X}$, for its j -th payment, $\forall j \in \{1, \dots, M_X^i\}$, where $\sum_{j=1}^{M_X^i} \Delta X_j^i = 1$

ΔY_j^i : Amortization rate of floating rate bond i , $\forall i \in \mathcal{X}$, for its j -th payment, $\forall j \in \{1, \dots, M_Y^i\}$, where $\sum_{j=1}^{M_Y^i} \Delta Y_j^i = 1$

Risk factors.

$f_t(s)$: Cash flow at time $t \in \mathcal{H} \cup \{T\}$, under scenario $s \in \mathcal{S}$

$r_{t,\tau}(s)$: Annual effective yield to maturity τ , during period $t \in \mathcal{H}$, under scenario $s \in \mathcal{S}$

$\rho_t(s)$: Cash account return during period $t \in \mathcal{H}$, under scenario $s \in \mathcal{S}$

$\alpha_t^i(s)$: Coupon of fixed rate bond $i \in \mathcal{X}$ issued at time $t \in \mathcal{H}$, under scenario $s \in \mathcal{S}$

$\psi_{t,k}(s)$: Risk premium at time $t \in \mathcal{H}$, under scenario $s \in \mathcal{S}$ over the yield to maturity k

2.2. Decision variables

These sets of variables include implementable policies such as the amount issued for each bond and also auxiliary variables to describe the state of the firm, e.g., cash account, asset and debt values. Note that we implicitly assume the non-negativity constraints of decision variables if not specified.

$X_{t,j}^i(s)$: Outstanding face value at time $t + j$ of fixed rate bond $i \in \mathcal{X}$ issued at $t \in \mathcal{H}^*$, under scenario $s \in \mathcal{S}$, where $j \in \{0, \dots, \min(t, M_X^i - 1)\}$

$Y_{t,j}^i(s)$: Outstanding face value at time $t + j$ of floating rate bond $i \in \mathcal{Y}$ issued at $t \in \mathcal{H}^*$, under scenario $s \in \mathcal{S}$, where $j \in \{0, \dots, \min(t, M_Y^i - 1)\}$

$C_t(s)$: Cash savings at time $t \in \mathcal{H}$, under scenario $s \in \mathcal{S}$

$C_T^+(s)$: Positive part of terminal cash savings under scenario $s \in \mathcal{S}$

$C_T^-(s)$: Negative part of terminal cash savings under scenario $s \in \mathcal{S}$

$D_t(s)$: Debt value at time $t \in \mathcal{H}$, under scenario $s \in \mathcal{S}$

$\bar{D}_t(s)$: Debt value at time $t \in \mathcal{H}$, under scenario $s \in \mathcal{S}$ excluding current issued bonds

$A_t(s)$: Asset value at time $t \in \mathcal{H}$, under scenario $s \in \mathcal{S}$, where

$$A_t(s) \in \mathbb{R}, \forall t \in \mathcal{H}, s \in \mathcal{S}$$

$I_{t,k}(s)$: Excess leverage at time $t \in \mathcal{H}$, scenario $s \in \mathcal{S}$, for the leverage limit $k = 1, \dots, K$

2.3. Balance constraints

Amortization. These constraints update the outstanding face value of each bond after amortization payments. For each bond i issued at t under scenario s , the outstanding face value at $t + j$ is the outstanding value at $t + j - 1$ minus the j -th amortization payment.

For fixed rate bonds, we have

$$\begin{aligned} X_{t,j}^i(s) &= X_{t,j-1}^i(s) - \Delta X_j^i \cdot X_{t,0}^i(s), \quad \forall i \in \mathcal{X} \setminus \{1\} \\ &\quad \forall t \in \mathcal{H}^*, \forall s \in \mathcal{S} \\ &\quad \forall j \in \{1, \dots, M_X^i - 1\}. \end{aligned}$$

For floating rate bonds, we have

$$\begin{aligned} Y_{t,j}^i(s) &= Y_{t,j-1}^i(s) - \Delta Y_j^i \cdot Y_{t,0}^i(s), \quad \forall i \in \mathcal{Y} \\ &\quad \forall t \in \mathcal{H}^*, \forall s \in \mathcal{S} \\ &\quad \forall j \in \{1, \dots, M_Y^i - 1\}. \end{aligned}$$

Cash balance. Cash balance constraints keep track of inflows and outflows at every stage of the system. We define them differently for the four portions of the planning horizon.

For $t = 0, \forall s \in \mathcal{S}$,

$$C_t(s) = c + f_t(s) - x_t - y_t \rho_{t-1}(s) - \Delta y_t + \sum_{i \in \mathcal{X}} X_{t,0}^i(s) + \sum_{i \in \mathcal{Y}} Y_{t,0}^i(s).$$

Total cash at the end of the first period is the initial value updated with new cash flow, minus payments and amortization for pre-existing debt, plus borrowing income, all at $t = 0$. Observe that the total current issuance is composed of two summations, adding all types of fixed and floating rate bond.

For $t \in \mathcal{H}^* \setminus \{0\}, \forall s \in \mathcal{S}$,

$$\begin{aligned}
 C_t(s) &= (1 + \rho_{t-1}(s))C_{t-1}(s) + f_t(s) - x_t - (y_t\rho_{t-1}(s) + \Delta y_t) \\
 &+ \sum_{i \in \mathcal{X}} X_{t,0}^i(s) + \sum_{i \in \mathcal{Y}} Y_{t,0}^i(s) \\
 &- \sum_{i \in \mathcal{X}} \sum_{j=1}^{\min(t, M_X^i)} (\alpha_{t-j}^i(s) X_{t-j, j-1}^i(s) + \Delta X_j^i \cdot X_{t-j,0}^i(s)) \\
 &- \sum_{i \in \mathcal{Y}} \sum_{j=1}^{\min(t, M_Y^i)} ((\rho_{t-1}(s) + \psi_{t-j, M_Y^i}(s)) Y_{t-j, j-1}^i(s) + \Delta Y_j^i \cdot Y_{t-j,0}^i(s)).
 \end{aligned}$$

As in $t = 0$, total cash is the previously accrued value updated with all current inflows and outflows, but also includes payments and amortization of new fixed and floating rate bonds.

For $t \in \mathcal{H} \setminus \mathcal{H}^*, \forall s \in \mathcal{S}$,

$$\begin{aligned}
 C_t(s) &= (1 + \rho_{t-1}(s))C_{t-1}(s) + f_t(s) - x_t - y_t\rho_{t-1}(s) + \Delta y_t \\
 &+ X_{t,0}^1(s) - (\alpha_{t-1}^1(s) X_{t-1,0}^1(s) + \Delta X_1^1 \cdot X_{t-1,0}^1(s)) \\
 &- \sum_{i \in \bar{\mathcal{X}}_t} \sum_{j=t-T^*+1}^{\min(t, M_X^i)} (\alpha_{t-j}^i(s) X_{t-j, j-1}^i(s) + \Delta X_j^i \cdot X_{t-j,0}^i(s)) \\
 &- \sum_{i \in \bar{\mathcal{Y}}_t} \sum_{j=t-T^*+1}^{\min(t, M_Y^i)} ((\rho_{t-1}(s) + \psi_{t-j, M_Y^i}(s)) Y_{t-j, j-1}^i(s) + \Delta Y_j^i \cdot Y_{t-j,0}^i(s)),
 \end{aligned}$$

where

$$\bar{\mathcal{X}}_t = \{i \mid i \in \mathcal{X} \setminus \{1\}, M_X^i \geq t - T^* + 1\},$$

$$\bar{\mathcal{Y}}_t = \{i \mid i \in \mathcal{Y}, M_Y^i \geq t - T^* + 1\}.$$

For the simplified portion of the horizon ($T^* \leq t < T$), cash balance constraints differ as new issuances are limited to short term bonds. Note also that summations limits in the terms corresponding to payments and amortization of long term bonds account for only those issued during the detailed horizon.

For $t = T, s \in \mathcal{S}$,

$$\begin{aligned} C_t^+(s) - C_t^-(s) &= (1 + \rho_{t-1}(s))C_{t-1}(s) + f_t(s) - x_t - y_t\rho_{t-1}(s) + \Delta y_t \\ &\quad - (\alpha_{t-1}^1(s)X_{t-1,0}^1(s) + \Delta X_1^1 \cdot X_{t-1,0}^1(s)) \\ &\quad - \sum_{i \in \bar{\mathcal{X}}_t} \sum_{j=t-T^*+1}^{\min(t, M_X^i)} (\alpha_{t-j}^i(s) \cdot X_{t-j, j-1}^i(s) + \Delta X_j^i \cdot X_{t-j,0}^i(s)) \\ &\quad - \sum_{i \in \bar{\mathcal{Y}}_t} \sum_{j=t-T^*+1}^{\min(t, M_Y^i)} ((\rho_{t-1}(s) + \psi_{t-j, M_Y^i}(s)) Y_{t-j, j-1}^i(s) + \Delta Y_j^i \cdot Y_{t-j,0}^i(s)). \end{aligned}$$

At the end of the planning horizon, we do not consider new debt issuance. Infeasibility is avoided by expressing the left hand side with two components and allowing for negative values in the final cash balance. Under scenario s , $C_T^+(s)$ represents the terminal cash savings, while $C_T^-(s)$ the outstanding obligations at the end of the horizon. We can also interpret $C_T^-(s)$ as the cash requirement to avoid insolvency. Note that we construct the objective function such that $C_T^+(s) C_T^-(s) = 0$.

Asset valuation. Net asset value at time t under scenario s is the conditional expectation of the present value of future project cash flows. Based on Miller and Modigliani [13, 14], we use the weighted average cost of capital (WACC) of the firm denoted by ω as the discount rate.

Then, for $t \in \mathcal{H}, s \in \mathcal{S}$:

$$A_t(s) = C_t(s) + \frac{1}{|\mathcal{S}(t, s)|} \sum_{\bar{s} \in \mathcal{S}(t, s)} \sum_{k=1}^{T-t} \frac{f_{t+k}(\bar{s})}{(1 + \omega)^k},$$

where $\mathcal{S}(t, s) = \{\bar{s} \in \mathcal{S} \mid N(t, s) = N(t, \bar{s})\}$. Note that \mathcal{S} stands for $\mathcal{S}(0, s), \forall s \in \mathcal{S}$.

Debt valuation. The market value of total outstanding debt at time t under scenario s is defined as the face value of current bond issues plus the market value of the previously issued debt. For fixed rate bonds, the market value is the net present value of their payments, discounted by the interest rate associated with each instrument. The outstanding face value defines the marked value for previously issued floating rate bonds.

For $t \in \mathcal{H}^*, s \in \mathcal{S}$,

$$D_t(s) = \sum_{i \in \mathcal{X}} X_{t,0}^i(s) + \sum_{i \in \mathcal{Y}} Y_{t,0}^i(s) + \tilde{D}_t(s)$$

where, for $t = 0$,

$$\tilde{D}_t(s) = \sum_{k=1}^{T-t} \frac{x_{t+k}}{(1 + r_{t,k}(s))^k} + y_t$$

while for $t \in \mathcal{H}^* \setminus \{0\}$,

$$\begin{aligned} \tilde{D}_t(s) = & \sum_{i \in \mathcal{X} \setminus \{1\}} \sum_{k=1}^{(M_X^i-1)} \sum_{j=k+1}^{\min(t+k, M_X^i)} \frac{\alpha_{t+k-j}^i(s) X_{t+k-j, j-1}^i(s) + \Delta X_j^i X_{t+k-j, 0}^i(s)}{(1 + r_{t,k}(s))^k} \\ & + \sum_{k=1}^{T-t} \frac{x_{t+k}}{(1 + r_{t,k}(s))^k} + \left(\sum_{i \in \mathcal{Y}} \sum_{j=1}^{\min(t, M_Y^i-1)} Y_{t-j, j}^i(s) \right) + y_t \end{aligned}$$

As in the cash balance constraints, for the detailed horizon, the value of currently issued bonds are computed as summations over all types of fixed and floating rate instruments. The value of previously issued debt, $\tilde{D}_t(s)$, has different definitions for the initial stage and the remainder of the detailed horizon.

For the simplified horizon, the total debt value is the currently issued short term bond, plus the market value of the all other instruments issued during the detailed horizon.

Then, for $t \in \mathcal{H} \setminus \mathcal{H}^*$, $s \in \mathcal{S}$,

$$D_t(s) = X_{t,0}^1(s) + \tilde{D}_t(s)$$

where,

$$\begin{aligned} \tilde{D}_t(s) = & \sum_{i \in \tilde{\mathcal{X}}_t} \sum_{k=1}^{(M_X^i-1)} \sum_{j=t+k-T^*+1}^{\min(t+k, M_X^i)} \frac{\alpha_{t+k-j}^i(s) X_{t+k-j, j-1}^i(s) + \Delta X_j^i X_{t+k-j, 0}^i(s)}{(1 + r_{t,k}(s))^k} \\ & + \sum_{k=1}^{T-t} \frac{x_{t+k}}{(1 + r_{t,k}(s))^k} + \left(\sum_{i \in \tilde{\mathcal{Y}}_t} \sum_{j=t-T^*+1}^{\min(t, M_Y^i-1)} Y_{t-j, j}^i(s) \right) + y_t \end{aligned}$$

and

$$\tilde{\mathcal{X}}_t = \{i \mid i \in \mathcal{X}, M_X^i \geq t - T^* + 2\},$$

$$\tilde{\mathcal{Y}}_t = \{i \mid i \in \mathcal{Y}, M_Y^i \geq t - T^* + 2\}.$$

Non-anticipativity. Thus far, our model formulation described only the relationships of decision variables within each scenario. The non-anticipativity constraints preserve the dynamic structure of the model by stating the equality of variables across different scenarios when they share the same history, or, equivalently, are associated with the same node in the event tree. This guarantees implementable optimal policies, where it is possible to state the corresponding dynamic programming equations. First, we define \mathcal{N} as the set of nodes in the tree and function $N(t, s) : \mathcal{H} \times \mathcal{S} \rightarrow \mathcal{N}$, mapping stage t in scenario s into its corresponding node. Then, we define the subsets of decision variable indexes for each node, $\mathcal{U}_n = \{(t, s) \mid N(t, s) = n\} \forall n \in \mathcal{N}$. For each non-singleton subset \mathcal{U}_n , we select a canonical element (t^*, s^*) and build the equality constraints linking corresponding decision variables with their counterparts associated with the other elements of the set.

For $n^* \in \{n \in \mathcal{N} \mid |\mathcal{U}_n| > 1\}$,

$$X_{t^*,0}^i(s^*) = X_{t,0}^i(s), \forall i \in \mathcal{X}, \forall (t, s) \in \mathcal{U}_{n^*}/(t^*, s^*),$$

$$Y_{t^*,0}^i(s^*) = Y_{t,0}^i(s), \forall i \in \mathcal{Y}, \forall (t, s) \in \mathcal{U}_{n^*}/(t^*, s^*).$$

Note that it is sufficient to consider only the constraints corresponding to $X_{t,0}^i(s)$ and $Y_{t,0}^i(s)$ since all other decision variable are consequently determined.

2.4. Objective function

The objective function in our model includes two contrasting components. The first measures the mean-risk trade-off between expected terminal cash savings and risk of default at the end of the horizon, expressed by a utility function on the terminal cash. For a risk neutral agent, since all accrued borrowing costs are accounted in the cash balance, we can easily establish that maximizing the expected terminal cash is equivalent to minimizing the expected future cost of servicing the debt. The latter quantity is commonly used as part of the objective in the debt management literature [1, 4, 5], combined with Conditional Value-at-Risk (CVaR) [17] as a risk aversion measure to be minimized or constrained. As shown in Shapiro [20], CVaR is not time consistent. Akin to non-anticipativity which forces identical decisions for scenarios sharing the same past, time consistency requires that optimality and feasibility should not depend on future scenarios that cannot happen when conditioned by the state at the moment of the decision. We argue that CVaR is inappropriate as a risk aversion measure for dynamic multistage stochastic programming models. As a matter of fact, including CVaR in the objective function may lead to suboptimality of the first stage decisions as illustrated in Rudloff et al. [18].

The second component of the objective function takes into consideration the company's debt worthiness based on financial performance measures available to market agents. Ideally, we would have included in the model an adjustment in interest rates reflecting the company's credit rating. However not only estimation of these corrections would not be possible with the available data, but it would greatly increase complexity, prohibitive even in moderately sized instances of MSP models. We propose instead a practical approach where a penalty function increasingly discourages excess leverage at intermediate stages of the planning horizon.

2.4.1. Terminal cash utility function

The utility function $U(C_T)$ assigns a value to a scenario at the end of the horizon based of the final cash balance. For the sake of ease in economic interpretation, we propose a piecewise linear function

$$U(C_T) = C_T^+ - p C_T^-,$$

where a negative terminal cash value is penalized by the risk aversion parameter $p \geq 1$. The expected value of the utility function is

$$\mathbb{E}[U(C_T)] = \mathbb{E}[C_T^+] - p \mathbb{E}[C_T^-],$$

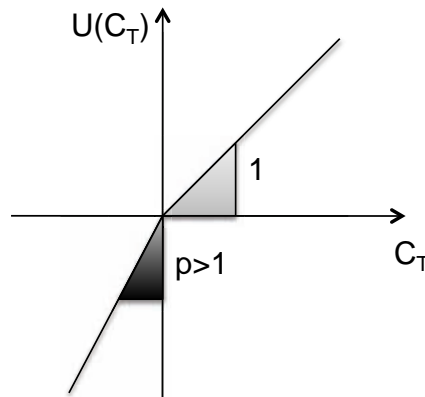


Figure 4: Terminal cash utility function

combining the expected terminal cash savings with the penalized expected value of insolvency.

This approach for measuring risk aversion is closely related to integrated chance constraints, which have also been used in financial planning problems [8], in particular in ALM models [9]. Observe in Figure 4 that coefficient p is a risk aversion parameter, with $p = 1$ representing a risk neutral agent.

2.4.2. Excess leverage penalty

The second part of our objective function deals with the effect of market perception in a company's bond issuance policy. As recommended by Lewellen and Emery [12] in a comparison of corporate debt management policies, firms should manage their debt by following a target on the Debt-to-Asset ratio. This ratio is also used frequently by market analysts as an indicator of the company's financial performance. Given this background, our model guides the optimal policies by including in the objective a penalty for high leverage debt positions. We propose a piecewise linear function that increasingly penalizes the excess leverage based on a sequence of threshold values for the Debt-to-Asset ratio. Denoted by $\gamma_1 \leq \dots \leq \gamma_K$, these values correspond to critical leverage levels as established by debt managers. In the objective function, we impose a cumulative penalty for violating each one of the leverage levels in each scenario, at each time period. First, we define the amount of excess leverage above each critical leverage level,

$$I_{t,k}(s) = [D_t(s) - \gamma_k A_t(s)]^+ = \max(0, D_t(s) - \gamma_k A_t(s)), \quad \forall t \in \mathcal{H}, s \in \mathcal{S}, k \in \mathcal{K}.$$

In the linear programming formulation of the model, this last expression is stated by initially adding as constraints the following inequalities,

$$I_{t,k}(s) \geq 0, I_{t,k}(s) \geq D_t(s) - \gamma_k A_t(s), \forall t \in \mathcal{H}, s \in \mathcal{S}, k \in \mathcal{K}.$$

The equality in the definition of variables $I_{t,k}(s)$ is guaranteed only in the optimal solution from the construction of the objective function which includes a penalty on each excess leverage value,

$$\theta_k I_{t,k}(s) \forall t \in \mathcal{H}, s \in \mathcal{S}, k \in \mathcal{K},$$

where $\theta_1 \leq \dots \leq \theta_K$ are positive penalty factors also assigned by debt managers.

The excess leverage penalty computes the total future values for each critical level violation in all scenarios,

$$\sum_{k \in \mathcal{K}} \theta_k \sum_{t \in \mathcal{H}} I_{t,k}(s) \prod_{\tau=t+1}^T (1 + \rho_\tau(s)) \forall s \in \mathcal{S}.$$

In its final form, we state the full objective stated by taking the expected values over all scenarios,

$$\max_{s \in \mathcal{S}} S^{-1} \sum_{s \in \mathcal{S}} (C_T^+(s) - pC_T^-(s) - \sum_{k \in \mathcal{K}} \theta_k \sum_{t \in \mathcal{H}} I_{t,k}(s) \prod_{\tau=t+1}^T (1 + \rho_\tau(s))),$$

noting that the penalty functions take negative signs in the maximization objective.

3. Illustrative example

In this section, we illustrate some key features of the model by building a simplified example where uncertainty is considered only on the term structure of the interest rates. We assume a project portfolio generating the deterministic cash flow stream given by Figure 5. With a planning horizon $T = 15$, the deterministic flows are repeated for all scenarios, setting the values for risk factors $f_i(s)$. We also assume a null return for the cash account, with $\rho_t(s) = 0$ for all time periods and scenarios.

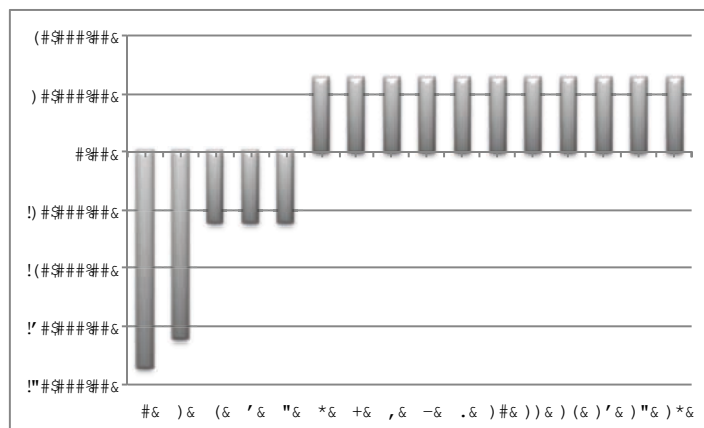


Figure 5: Deterministic cash flows for an illustrative project

Based on this uncertainty framework, the model is further specified by the portfolio of available debt instruments and the generation of the event tree. The resulting problem is solved for the maximization of the terminal cash utility only. When compared to a complete instance of the implemented model, this example allows for a larger number of scenarios in the event tree. The low computational effort in the solution of each instance also permits building the efficient frontier for the risk aversion parameter.

3.1. Debt instruments

We consider nine types of bonds of varying maturities and amortization schedules:

Fixed Rate Bonds

Short Term: 1-year short-term bond

Fixed-5 Final: 5-year bond with full amortization at the end

Fixed-10 Final: 10-year bond with full amortization at the end

Fixed-5 Constant: 5-year bond with constant amortization bond

Fixed-10 Constant: 10-year bond with constant amortization bond

Floating Rate Bonds

Floating-5 Final: 5-year bond with full amortization at the end

Floating-10 Final: 10-year bond with full amortization at the end

Floating-5 Constant: 5-year bond with constant amortization

Floating-10 Constant: 10-year bond with constant amortization

Expressing these definitions as parameters of the model, we have $nY = 4$ and $nX = 5$. Note that the set of fixed-rate bonds include short-term instruments indexed by $i = 1$ in $X = \{1, 2, 3, 4, 5\}$. Amortization schedules are defined as:

For $i \in \{1, 2, 3\}$,

$$\Delta X_j^i = \begin{cases} 1, & \text{for } j = M_X^i \\ 0, & \text{otherwise} \end{cases}$$

For $i \in \{4, 5\}$,

$$\Delta X_j^i = \frac{1}{M_X^i}, \forall j = 1, \dots, M_X^i.$$

Maturities are defined as:

$$M_X^i = \begin{cases} 1, & \text{for } i = 1 \\ 5, & \text{for } i \in \{2, 4\} \\ 10, & \text{for } i \in \{3, 5\} \end{cases}$$

For the floating rate bonds, we have $\mathcal{Y} = \{1, 2, 3, 4\}$, with the amortization schedules defined as:

For $i \in \{1, 2\}$,

$$\Delta Y_j^i = \begin{cases} 1, & \text{for } j = M_Y^i \\ 0, & \text{otherwise} \end{cases}$$

For $i \in \{3, 4\}$,

$$\Delta Y_j^i = \frac{1}{M_Y^i}, \quad \forall j = 1, \dots, M_X^i.$$

Maturities are defined as follows:

$$M_Y^i = \begin{cases} 1, & \text{for } i = 1 \\ 5, & \text{for } i \in \{2, 4\} \\ 10, & \text{for } i \in \{3, 5\} \end{cases}$$

3.2. Scenario tree generation

Scenarios for the MSP are generated by the forecasting model presented in Vereda [21], which supplies the estimated parameters for Brazilian and American term structure of the interest rates in the following state space framework:

$$\begin{aligned} \eta_t &= A + B\xi_t \\ \xi_t &= \Phi\xi_{t-1} + \Sigma^{1/2}\varepsilon_t, \quad \varepsilon_t \sim N(0, I). \end{aligned}$$

Based on the *Adjusted Random Sampling* of Kouwenberg [11], we compute an event tree that approximates the original stochastic vector ε_t , using antithetic values along with a variance adjustment. For the sake of implementation efficiency, we generate the residual tree nodewise, i.e., $\varepsilon(n)$, $\forall n \in \mathcal{N}$.

Given a node n , let us denote $\mathcal{Q}(n) = \{q_1, q_2, \dots\}$ the set of all possible successor nodes, with $Q(n) = |\mathcal{Q}(n)|$. Using antithetic values, we match the zero mean and all null higher odd moments for each univariate stochastic component of $\varepsilon_i(q)$, $\forall q \in \mathcal{Q}(n)$. After initializing $\varepsilon_i(q) = 0$, $\forall q \in \mathcal{Q}(n)$, we sample via Monte Carlo simulation the first $k = 1, \dots, \lfloor Q(n)/2 \rfloor$ elements and generate the antithetic values for the remainder for $j = Q(n) - k$,

$$\varepsilon(q_j) = -\varepsilon(q_k).$$

Note that this procedure ensures null conditional odd moments of the simulated $\varepsilon_i(n)$ for each component i , i.e.,

$$\mathbb{E}[\varepsilon_i^p | n \in \mathcal{N}] = \frac{1}{Q(n)} \sum_{q \in \mathcal{Q}(n)} \varepsilon_i^p(q) = 0, \quad \forall p = 1, 3, 5, \dots$$

Returning to the original notation, we define the unadjusted residuals as

$$\tilde{\varepsilon}_i(s) = \varepsilon_i(n), \quad \forall t \in \mathcal{H} \cup \{T\}, \quad s \in \mathcal{S} \text{ such that } n = N(t, s).$$

Then, we adjust the variance of $\tilde{\varepsilon}_{i,t}$ for each stage t and for each component i . Indeed, to suit the hybrid tree structure, our procedure matches the unconditional variances in opposition to the conditional approach of Kouwenberg [11]. Therefore the adjusted residuals are given by

$$\varepsilon_{i,t}(s) = \frac{\tilde{\varepsilon}_{i,t}(s)}{\sqrt{\frac{1}{S} \sum_{s \in \mathcal{S}} \tilde{\varepsilon}_{i,t}^2(s)}}, \quad \forall s \in \mathcal{S}.$$

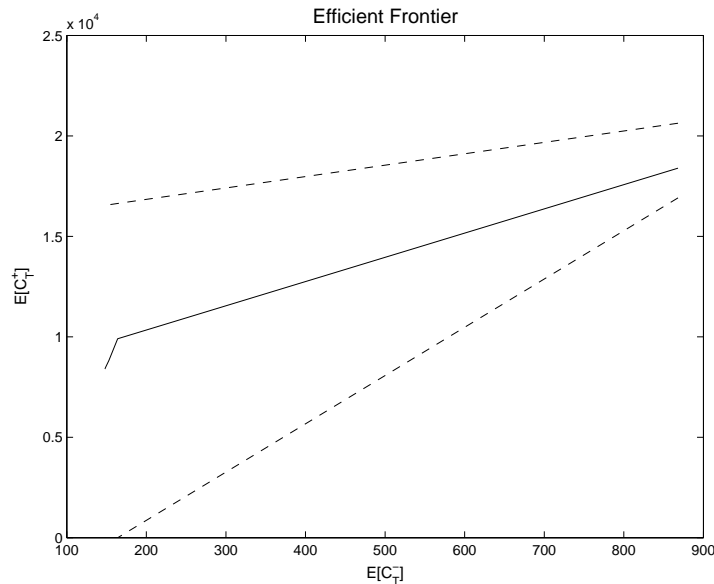


Figure 6: Efficient Frontier

3.3. Solution and Sensitivity analysis

Sensitivity analysis examines the robustness and stability of the optimal solution vis-à-vis changes in the input parameters and data uncertainty. With the objective function limited to the terminal cash utility function, without the excess leverage penalty, this experiment builds the efficient frontier for the risk aversion parameter p . Since the proposed model is only tractable for relatively small event trees, the optimal solution is subject to estimation errors. Given a value for p and the methodology described in 3.2, we generate N independent event trees whose sets of scenarios are denoted by $\mathcal{S}_i, \forall i = 1, \dots, N$. Then, we solve the problem for each scenario set, using the optimal solution to compute the two components of the objective function, $C_i^+ = S^{-1} \sum_{s \in \mathcal{S}_i} C_T^+(s)$ and $C_i^- = S^{-1} \sum_{s \in \mathcal{S}_i} C_T^-(s), \forall i = 1, \dots, N$.

Ultimately, we build the efficient frontier corresponding to each scenario set \mathcal{S}_i by solving the problem for each value of p and linearly interpolating the observed points to compute the curve C_i^+ vs $C_i^-, \forall i \in \{1, \dots, N\}$. In our experiment, we assumed $N = 1000$ and $p \in \{1, 50, 100, 200, 500, 1000, 2000\}$, generating 7000 instances of the model. The resulting efficient frontier is represented in Figure 6, where the average and 95% percentile are obtained from the distribution of all possible values of C^+ given a fixed level of risk C^- .

From the results of this experiment, we can also develop a sensitivity analysis for the first stage decision with respect to the risk aversion parameter p . For each $i = 1, \dots, N$, we take the first stage optimal solution for each scenario set $\mathcal{S}_i, Z_i = (X_{0,0}^1, \dots, X_{0,0}^{nX}, Y_{0,0}^1, \dots, Y_{0,0}^{nY})$, indicating the amounts issued in fixed and floating rate bonds, for all maturities and amortization schedules. Then, we compute the sample

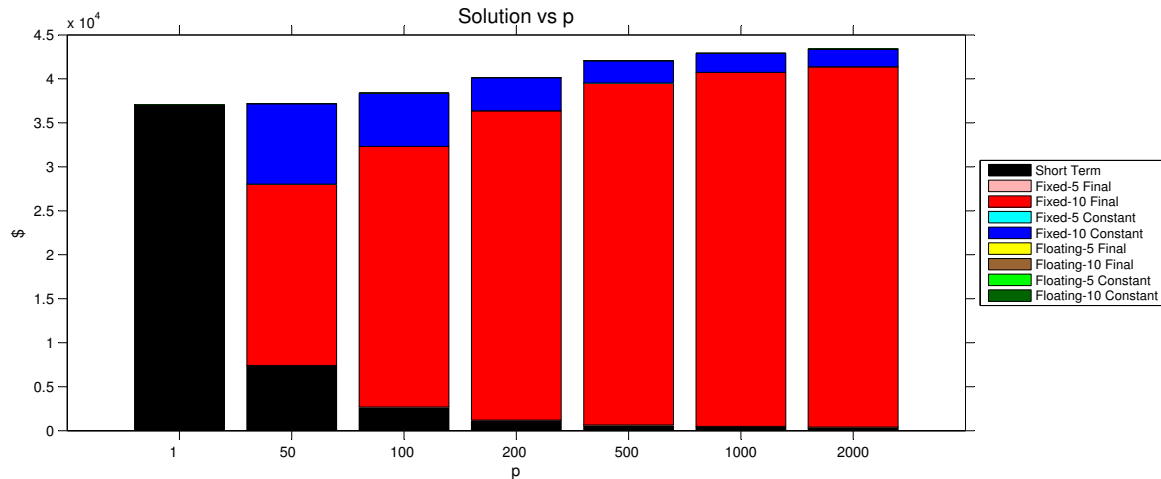


Figure 7: First Stage Decision and Risk Aversion

average approximation $\bar{Z} = N^{-1} \sum_{i=1}^N Z_i$ for all values of p . The stacked bar graph in Figure 7 indicates the amounts corresponding to each available debt instrument, expressing the behavior of the optimal solution with respect to risk aversion level. Note that for the risk neutral case, $p = 1$, the short term bond is preferred while the risk averse case, $p > 1$, long term bonds are increasingly more attractive. The explanation for this behavior is that the cost of long term debt is locked until maturity, while issuing a short term portfolio in the first stage decision involves refinancing in the future, subject to uncertainty in borrowing cost and hence higher risk.

4. Application to the oil industry

Application of the proposed model to a real-world problem requires further assumptions, complete specification of risk factors and inclusion of the excess leverage penalty into the objective function. These features were implemented in a financial planning software tool deployed in a risk management organization at Brazilian oil company Petrobras. Two auxiliary modules have been developed for the generation of risk factor scenarios: An integrated interest and exchange rates forecasting model and a simulator for future spot prices of crude oil, its byproducts and natural gas. Considering the same debt instruments as before, we specialize the various elements of our multistage stochastic programming model for this application and present results based on a fictitious, although realistic, project data set.

4.1. Risk factors

The model formulation constraints refer explicitly to scenarios for interest rates and risk premiums. We call those *Financial Risk Factors*. In addition, *Project Risk Factors* are reflected indirectly in the scenarios for cash flows generated by the project portfolio, including market prices for crude oil and natural gas. In an attempt to approximate

continuous flow of revenues and expenditures distributed over each planning stage, we assume the average values during the period. The detailed description of the forecast models is available in [21] and [7].

Financial Risk Factors. An integrated forecast model provides scenarios for interest and exchange rates.

$r_{t,\tau}(s)$: Annual effective yield for bonds denominated in US\$, issued by the company

$\rho_t(s)$: Risk free interest rate, assumed to be US government bond 1-year yield

$\psi_{t,k}(s)$: Risk premium associated with the company for the corporate bonds denominated in US\$

Note that there is a unique mapping between the fixed rate coupons and the term structure of the interest rate. Therefore, $\alpha_t^i(s)$ must be derived from the corresponding term structure $r_{t,j}(s)$, $\forall j = 1, \dots, M_X^i$, assuming the net present value of future payments is equal to its face value. Without loss of generality, we compute the coupon using a unit face value, i.e.,

$$1 = \alpha_t^i(s) \left(\frac{1}{1 + r_{t,1}(s)} + \sum_{j=2}^{M_X^i} \frac{1 - \sum_{k=1}^{j-1} \Delta X_k^i}{(1 + r_{t,j}(s))^j} \right) + \sum_{j=1}^{M_X^i} \frac{\Delta X_j^i}{(1 + r_{t,j}(s))^j}.$$

Then, the coupon is defined as

$$\alpha_t^i(s) = \left(1 - \sum_{j=1}^{M_X^i} \frac{\Delta X_j^i}{(1 + r_{t,j}(s))^j} \right) \left(\frac{1}{1 + r_{t,1}(s)} + \sum_{j=2}^{M_X^i} \frac{1 - \sum_{k=1}^{j-1} \Delta X_k^i}{(1 + r_{t,j}(s))^j} \right)^{-1}.$$

Project Risk Factors. We assume all stochastic cash flows generated by the project portfolios to be an affine functions of project risk factors. Besides market prices for crude oil and its byproducts, exchange rates are also considered risk factors in this category, as the project portfolio includes multi-currency investments. Based on these risk factors, investments and production data, a preprocessor to the optimization model computes scenarios for the cash streams $f_t(s)$, $d_t(s)$ and $l_t(s)$.

4.2. Computational experiments

The financial planning software tool implemented from our model uses Matlab to perform all of the data preparation and solution presentation. The linear programming formulation was implemented with the MOSEL modeling language, using the Xpress optimization suite as the solver. The computational experiments were carried out on an Intel(R) Core(TM) i7 CPU based computer with 24Gb RAM and 8 processors.

As opposed to the illustrative example presented in Section 3 where we assume a null return for the cash account, savings are now invested in short term US government bonds subject to stochastic returns. This key assumption closely matches the actual corporate financial strategy. However, it increases the probability of outlier scenarios where the income generated by the cash account is greater than the costs of some

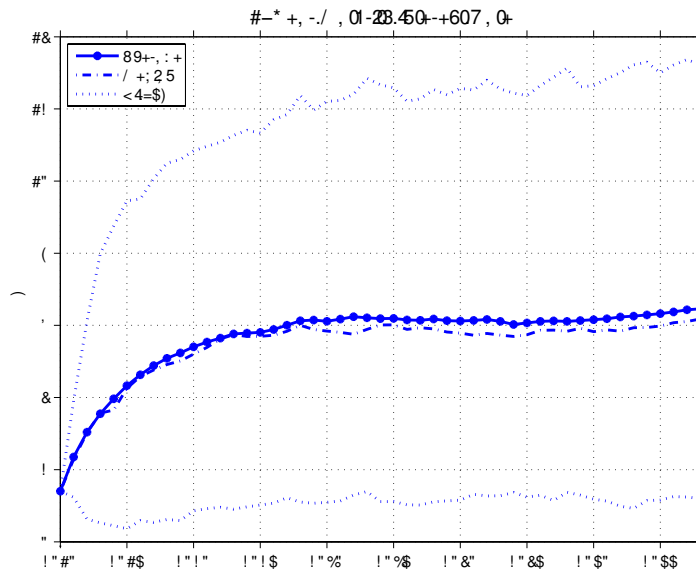


Figure 8: Short Term Stochastic Interest Rate (1-year maturity)

c	Initial cash savings	\$18.00
\hat{c}_t	Minimum cash ($\forall t \in \mathcal{H}$)	\$5.00
p	Risk aversion parameter for insolvent scenarios	2
ω	Weighted average cost of capital (WACC)	8.8%
K	Number of leverage rate segments	3

Table 1: Base case parameter values.

debt instruments. The optimal policy for these scenarios would be a highly leveraged, possibly unbounded, debt portfolio. Under these circumstances, intermediate excess leverage penalties are added to the objective function, avoiding unrealistic solutions. We have chosen the specific set of interest rate scenarios illustrated in Figure 8 and Figure 9 to emphasize the effect of our multi-criteria objective function.

In this experiment, we consider a base case with a 48-year horizon ($T = 48$) starting from 2010, a 6-year detailed horizon ($T^* = 6$), 1024 scenarios and the set of parameters defined in Table 1. With this specification, the resulting equivalent deterministic linear program has 820534 rows, 813056 columns and 4559468 non-zero matrix elements. We compare the solutions of the problem under two assumptions for the excess leverage penalties $\{\theta_k\}_{k=1}^K$. As presented in Table 4.2, in Case 1 we assume zero penalties and an arithmetic progression in Case 2.

Examining the optimal solutions for both cases, we first compare the expected bond issuances for the detailed portion of the horizon. Figure 10 shows the average amount issued for each bond on each stage $t \in \mathcal{H}^*$. For Case 1, the total debt issued is much higher than the amount required to fulfill the minimum cash requirement of the firm,

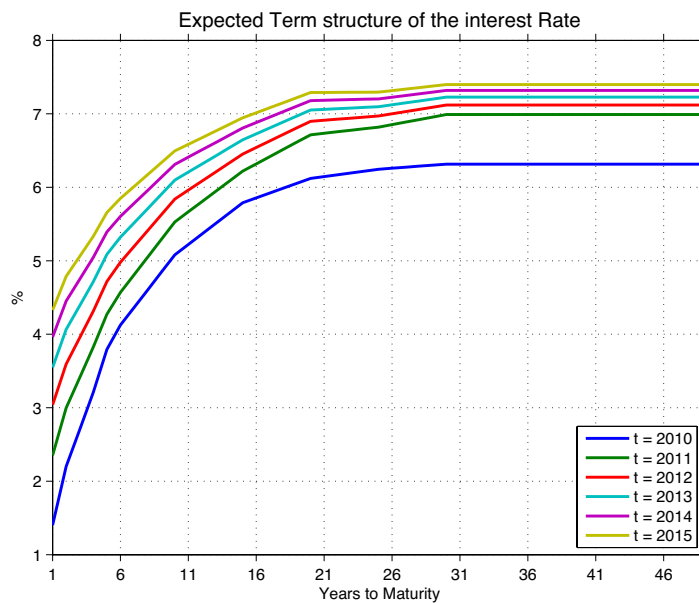


Figure 9: Expected Term Structure of the Interest Rate

Index (k)	Break point (γ_k)	Additional Penalty (θ_k)	
		Case 1	Case 2
1	35%	0	1
2	50%	0	1
3	100%	0	1

Table 2: Leverage parameter values.

indicating the presence of scenarios with cash saving earnings above debt costs. This effect disappears in Case 2.

The impact of imposing intermediate penalties is further illustrated by analyzing the behavior of the stochastic leverage ratio, $D_t(s)/A_t(s)$. Figure 11 displays in different colors the probability that the leverage ratio belongs to each range of target values, noting that all scenarios have the same solution in the first stage.

The solution for Case 1 counters the intuitive premise that a firm with a fixed project portfolio should not be unnecessarily exposed to risk from uncertain financial returns. In Case 2, the intermediate penalties discourage risky policies with high leverage ratios occurring in a small subset of scenarios. The last leverage range, $D_t(s)/A_t(s) \geq 100\%$, defines the insolvency state at each stage, when debt exceeds total assets. As anticipated, the leverage penalty also reduces the insolvency probability at intermediate stages. This experiment suggests that debt managers use the proposed model interactively, tuning risk aversion parameters in the intermediate excess leverage penalties to avoid highly leveraged portfolios.

5. Conclusion

We propose a multistage linear stochastic programming model for optimal bond issuance of a firm considering fixed and floating interest rate bonds with different maturities and amortization patterns.

We proposed an approximation for a numerically intractable long term multistage problem. We assume a hybrid model where the first T^* stages are represented by a full event tree and the remainder described by subsamples approximated by independent paths. For the first period, we considered a full bond portfolio, while the simplified period considers only short term debt. Nonetheless suboptimal, we argue that our solution is a good approximation when most of the expenses are due over the first stages.

Moreover, we proposed a objective function as the expected utility function that minimizes the cost of funding and penalizes negative values for the cash account at the terminal stage. In addition, we include intermediate excess of leverage penalties considering the market values of assets and debt. To do so, we develop valuation methods within the stochastic programming and considering a convex piecewise linear penalty with break points related to threshold levels of leverage. We examined the behavior of the model in an illustrative example that evidences the importance of this penalty function to introduce an appropriate risk aversion to the model.

To sum up, this work develops a corporate debt management model via MSP to handle multiplicity of bond characteristics while minimizing expected costs, risks and performance measures. By virtue of the dimensionality curse of MSP models, we introduce an approximation for the information structure of uncertainty represented by a detailed event tree in addition to a subsample of independent scenarios. Indeed, we argue that this representation is a good approximation for a set of projects where most investments due on the detailed period.

Acknowledgments

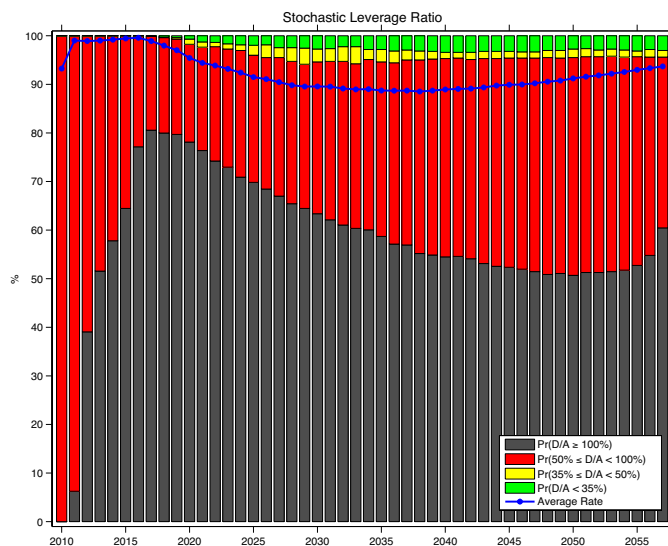
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Case 1: No intermediate penalties



Case 2: Additive intermediate penalties

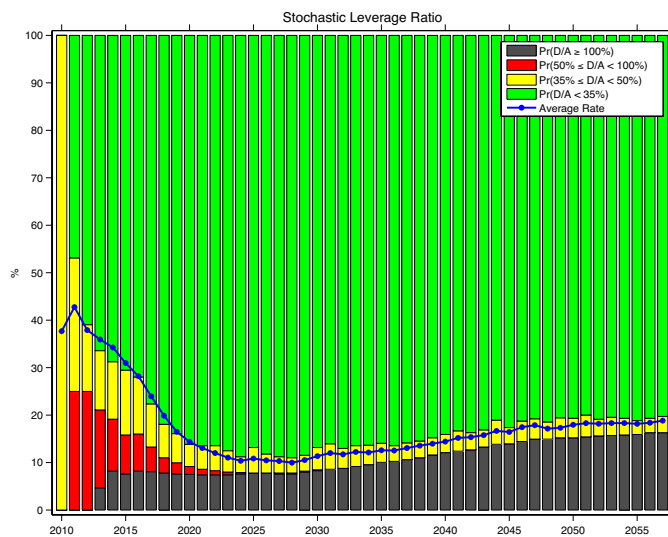


Figure 11: Stochastic Leverage Ratio