THE AIR TRANSPORTATION HUB-AND-SPOKE DESIGN PROBLEM: COMPARISON BETWEEN A CONTINUOUS AND A DISCRETE SOLUTION METHOD

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ABSTRACT.

Hub-and-spoke network designs arise in logistics systems in which flows do exist among distinct points and where economy of scale can be attained through concentration points and through the shared used of high capacity links. The hub-and-spoke network design problem, also known as the hub location problem, aims to find the concentration points in a given network flow so that the sum of the distances of the linkages is minimized. In this work, we compare the discrete solutions given by the branch-and-cut method applied to the p-hub median model, with the continuous ones given by the hyperbolic smoothing technique applied to a min-sum-min model. Computational experiments for particular instances of the Brazilian air transportation system, with the number of hubs varying from 2 to 6, are conducted with the support of the Voronoi diagrams.
Keywords: hub location problem, p-median, hyperbolic smoothing technique

RESUMO.

Projetos de rede do tipo *hub-and-spoke* surgem em sistemas logísticos em que existem fluxos entre pontos distintos e onde economia de escala pode ser alcançada através da concentração de pontos e do uso de ligações compartilhadas com alta capacidade. O problema de projeto de rede *hub-and-spoke*, também conhecido como problema de localização de hubs, visa encontrar pontos de concentração numa determinada rede de fluxos de modo que a soma das distâncias das ligações seja minimizada. Neste trabalho, comparamos as soluções discretas obtidas pelo método de *branch-and-cut* aplicadado ao modelo da p-hub mediana com as soluções contínuas obtidas pela técnica de suavização hiperbólica aplicada num modelo min-sum-min. Experimentos computacionais para instâncias particulares do sistema de transporte aéreo brasileiro, com o número de *hubs* variando de 2 a 6, são realizados com auxílio dos diagramas de Voronoi.
Palavras-chaves: problema hub-and-spoke, modelo p-mediana, suavização hiperbólica

1 Introduction

Air freight has grown faster in the past few decades (Bowen, 2004). Particularly in Brazil, about 155 million trips were processed in 2010, 21% more than in the previous year (INFRAERO, 2010). This growth calls for a rationalization of the air transport system. In this context, there are challenges to be faced in serving this growing demand. The most important appears to be the poor airport infrastructure that prevails in Brazil, where the busiest airports face huge problems of congestion and most do not have physical space available for expansion. A useful strategy to improve the configuration of air transportation systems is the hub-and-spoke network design.

According to Fulco (2006), the word *hub* can be described as a center of importance or interest and the word *spoke* can be defined as a link or arc. Therefore, the hub-and-spoke system can be interpreted as relevant centers connected to many nodes through links. This strategy aims to optimize the number/distances of linkages in the network, diminishing the overall costs. In this sense, the hub-and-spoke network design problem, also known as the hub location problem, aims to find the centers that are linked to each other, which in turn concentrate regional interconnected nodes that enable minimization of the flow costs.

As in Pizzolato *et al.* (2012), the formal study of hub locations was introduced by O'Kelly (1987) who provides a quadratic programming model for the hub location problem, and proposed two enumeration-based heuristics to solve it. The modeling design considered as assumption that the number p of hubs is determined a priori, there is no limit on the number of spokes assigned to a hub and each spoke is assigned to a single hub and all hubs are interconnected.

Indeed, the hub location problem has attracted the attention of researchers from a wide variety of science fields, such as Geography, Operations Research, Transportation (of passengers and of cargo), Telecommunications, among others. An overview of the academic research on hub location problems can be found in Alumur & Kara (2008), Hekmalfar & Pishvace (2009) and Campbell *et al.* (2002), where it can be verified that many sophisticated models have been proposed.

In this preliminary study, we deal with the hub-and-spoke design of the Brazilian air transportation system, aiming to compare a discrete and a continuous solution method. To this end, the bi-dimensional hub location problem is treated in a simplest way, that is: the only considered parameters of the original problem are the geographical coordinates (latitude and longitude) of the Brazilian airports. No costs are considered in this study.

First, we solve the hub location problem by formulating it as the p-hub median model, using the same analogy made by Campbell (1996), where a demand point in a p-median model is analogous to an origin-destination pair in the hub-and-spoke problem. The resulting model is an integer linear programming problem whose instances are solved by the branch-and-cut method.

Then, we consider the application of the hyperbolic smoothing technique, introduced by Xavier (2001), to solve a min-sum-min continuous formulation that models the hub location problem. In general terms, this technique solves a sequence of low dimensional differentiable unconstrained continuous optimization subproblems whose solutions gradually approach a solution of the original problem. This method has been applied to solve other optimization problems such as the determination of spatial molecular structure (Souza *et al.*, 2011) and the determination of radio telecommunication bases (Brito & Xavier, 2006), for example.

We observe that the discrete method chooses as hub locations the points (nodes) over the network, whereas the solutions of the continuous method can be points out of the network (not nodes), but in between the given network points. So, in other to compare both methods in respect to the minimization of the spoke-distances, we apply a procedure to approximate the point outside the network, given by the continuous method, to the nearest point over the network that results in lower spoke-distances, whenever it is needed.

Computation experiments are conducted with a network of 41 airports of the Brazilian air transportation system, extract from the network of 135 airports built by Figueiredo *et al.* (2012). The solutions of both methods are compared, varying from 2 to 6 the number of hubs, with the support of the Voronoi diagrams.

The paper is organized as follows. As the hub location problem has been already introduced, in Section 2 we briefly describe the p-hub median model. Section 3 presents the hyperbolic smoothing technique which is applied to solve a min-sum-min continuous model. Section 4 presents the numerical experiments with both methods and with the Voronoi diagram, while Section 5 outlines the conclusions.

2 The p-hub median approach

Likewise Campbell (1996), we formulate the hub location problem as the p-median model, where a demand point is interpreted as an origin-destination pair of the hub-and-spoke network, in such a way that the model is now called the p-hub median. So, in this approach, the location problem seeks p-medians or p concentration points to locate hubs in order to minimize the system impedance, that is, the sum of the spoke-distances.

In the following, we briefly describe the p-hub median model. Let S be the set of n distinct points of a given network in the bi-dimensional space; $i \in S$ a given point; $j \in S$ a potential hub median; p the number of hubs to be located; $[d_{ij}]_{n\times n}$ the symmetric distance matrix, where d_{ij} is the distance from point i to potential hub j, with $d_{ii} = 0$, $\forall i$; $[x_{ij}]_{n\times n}$ the allocation matrix, where $x_{ij} = 1$ if point i is connected to hub j and $x_{ij} = 0$ otherwise; and $x_{jj} = 1$ if j is a hub median, and $x_{jj} = 0$, otherwise.

Assuming by hypothesis that any point can be chosen as hub, the p-hub median model is formulated as follows:

$$\text{minimize} \quad \sum_{i\in S}\sum_{j\in S} d_{ij}x_{ij} \tag{1}$$

$$\text{subject to} \quad \sum_{j\in S} x_{ij} = 1, \quad i \in S \tag{2}$$

$$\sum_{j\in S} x_{jj} = p \tag{3}$$

$$x_{ij} - x_{jj} \leq 0, \quad i,\ j \in S \tag{4}$$

$$x_{ij} \in \{0,1\}, \quad i,\ j \in S, \tag{5}$$

where the objective function (1) indicates the minimization of the total distances between the given points and the points selected to be hubs; the constraint set (2) indicates that each point i is connected to only one hub j; the constraint (3) guarantees that there exist exactly p located hubs; the constraints (4) state that a given point must be connected to a hub, if it is not a hub itself, and finally the constraint set (5) imposes binary decision variables.

Notice that instead of formulating the hub location problem the model (1)-(5) can be used to formulate the clustering problem, where the p clustering points cover/concentrate the remaining $n - p$ points.

3 The hyperbolic smoothing approach

Let $S = \{s_1, \dots, s_n\}$ denote the set of n given distinct points in the bi-dimensional space to be clustered into a given number p of unknown distinct points. Let x_i, $i = 1, \dots, p$, be the position of the p potential hubs. In this approach, the hub location problem is formulated as a continuous min-sum-min model as follows:

$$\begin{array}{ll} \text{mininize} & \sum_{j=1}^{n} z_j \\ \text{subject to} & z_j = \min_{i=1,\dots,p} \|s_j - x_i\|_2, \quad j = 1, \dots, n, \end{array} \tag{6}$$

where the objective function is the sum of each minimum distance between a given point and a potential hub.

So, in order to obtain a convex and differentiable version of the above model, we proceed with the hyperbolic smoothing technique, proposed by Xavier (2001), which applies a sequence of transformations to the above model. Let us first consider the relaxation of the constraints in (6), getting:

$$\begin{array}{ll} \text{minimize} & \sum_{j=1}^{n} z_j \\ \text{subject to} & z_j - \|s_j - x_i\|_2 \leq 0 \quad j = 1, \dots, n, \quad i = 1, \dots, p. \end{array} \tag{7}$$

Now, in order to obtain the desired equivalence between (6) and (7), the function $\psi : \mathbb{R} \to \mathbb{R}$, defined as $\psi(y) = max\{0, y\}$, is introduced into the inequalities, obtaining the new formulation:

$$\sum_{i=1}^{q} \psi(z_j - \|s_j - x_i\|_2) = 0, \quad j = 1, \dots, n. \tag{8}$$

With (8) instead of the constraint set in (7), we still have an undesirable formulation, since the problem has no lower bound. To overcome this shortcoming, a given perturbation parameter $\varepsilon > 0$ is introduced, resulting in the bounded but not differentiable optimization problem:

$$\begin{array}{ll} \text{minimize} & \sum_{j=1}^{n} z_j \\ \text{subject to} & \sum_{i=1}^{p} \psi(z_j - \|s_j - x_i\|_2) \geq \varepsilon, \quad j = 1, \dots, n. \end{array} \tag{9}$$

Now, given the parameter $\tau > 0$, let us consider the hyperbolic function $\phi : \mathbb{R} \to \mathbb{R}$, defined as $\phi(y, \tau) = \left(y + \sqrt{y^2 + \tau^2}\right)/2$. Thus, the smoothness of model (9) is obtained by replacing $\psi(y)$ by $\phi, (y, \tau)$, and also by replacing the norm function by the function $\theta : \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}_+ \to \mathbb{R}_+$, defined as $\theta(s_j, x_i, \gamma) = \sqrt{(s_j^1 - x_i^1)^2 + (s_j^2 - x_i^2)^2 + \gamma^2}$ for a given parameter $\gamma > 0$, which completes the smoothness procedure, and generates the following problem:

$$\begin{array}{ll} \text{minimize} & \sum_{j=1}^{n} z_j \\ \text{subject to} & \sum_{i=1}^{p} \phi(z_j - \theta(s_j, x_i, \gamma), \tau) \geq \varepsilon, \quad j = 1, \dots, n. \end{array} \tag{10}$$

As the constraints of problem (10) are monotonically increasing functions in the variable z_j, $j = 1, \ldots, n$, (Xavier, 2011), they will be active at the optimal solution, and thus problem (10) is equivalent to the following problem:

$$\begin{aligned} &\text{minimize} \quad \sum_{j=1}^{n} z_j \\ &\text{subject to} \quad h_j(x, z_j) = \sum_{i=1}^{p} \phi(z_j - \theta(s_j, x_i, \gamma), \tau) - \varepsilon = 0, \quad j = 1, \ldots, n. \end{aligned} \tag{11}$$

Observe that model (11) has a separable structure, since each auxiliary variable z_j appears only in one equality constraint. Therefore, as the partial derivative of $h(x, z_j)$ with respect to z_j, $j = 1, \ldots, n$, is not equal to zero, it is possible to apply the results of the Implicit Function Theorem to compute each component z_j, $j = 1, \ldots, n$, as a function of the hub variables x_i, $i = 1, \ldots, p$. In this way, problem (11) is rewritten as the unconstrained optimization problem

$$\text{minimize } f(x) = \sum_{j=1}^{n} z_j(x) \tag{12}$$

where each $z_j(x)$ results from the computation of the single zero of each constraint in (11), since in each sum term ϕ is strictly increasing with respect to variable z_j. From the Implicit Function Theorem, the functions $z_j(x)$ have derivatives with respect to the variables x_i, and so it is possible to compute the gradient of the objective function of problem (12) as

$$\nabla f(x) = \sum_{j=1}^{n} \nabla z_j(x)$$

where

$$\nabla z_j(x) = -\, \nabla h_j(x, z_j) \,/\, \left(\frac{\partial\, h_j(x, z_j)}{\partial\, z_j} \right).$$

Summing up, the solution of the hub location problem can be obtained by using an algorithm which solves an infinite sequence of continuous optimization subproblems (12), where the positive parameters ε, τ, and γ are gradually reduced to zero, just as a smoothing method. Notice that when the algorithm causes τ and γ to approach 0, the constraints of the subproblems given in (10) tend to those of (9). Also, when the algorithm causes ε to approach 0 simultaneously, the solution of problem (9) gradually approaches the solution of the original hub location problem (6). Additionally, each unconstrained subproblem (12) can be solved by any method based on first order derivative information.

4 Voronoi diagram technique

Proposed originally by Georgy Voronoi, in the middle of the nineteenth century, the Voronoi diagram is a special type of metric space partition determined by Euclidian distances from a set of specified seed points in the bi-dimensional space. This partition with at least 2 seed points results in convex bi-dimensional polygons such that each polygon contains exactly one of these seed points, as well as it contains interior points that are closer to this seed point than any other seed point.

Resembling Boots and South (1997), the ordinary Voronoi diagram can be defined as follows. Let $\{x_1, x_2, \ldots, x_p\}$ be the set of p known distinct seed points in $\mathbb{R}^2$. Let x denote an arbitrary point in $\mathbb{R}^2$. As the Euclidean distance between x and x_j is given by

$$\|x - x_j\| = \sqrt{(x^1 - x_j^1)^2 + (x^2 - x_j^2)^2},$$

the region $V(x_j) = \{x \in \mathbb{R}^2 \; : \; \|x - x_j\| \leq \|x - x_i\|, \; j \neq i, \; i, j = 1, \ldots, p\}$ is called the ordinary Voronoi polygon of the seed point x_j. Indeed, $V(x_j)$ contains all the points that are closer to the seed point x_j than any other. Moreover, the set $\Gamma(P) = \{V(x_1), \ldots, V(x_p)\}$ is a partition of $\mathbb{R}^2$, called the ordinary Voronoi diagram of the set of seed points.

5 Computational Experiments

In this section, we compare the continuous and the discrete solution approaches that address the hub location problem applied to the Brazilian air transportation system. We then present the computational results obtained by the application of the branch-and-cut algorithm to solve the p-hub median model (1)-(5) and by the application of the hyperbolic smoothing technique to solve the min-sum-min model (6). From solver CPLEX 11.2 we get the results of the branch-and-cut algorithm. The unconstrained minimization subproblems that are generated by the hyperbolic smoothing technique were coded with Compact Visual FORTRAN, version 6.1, and solved by the BFGS algorithm, a quasi-Newton method from the Harwell Library, available at http://www.cse.scitech.ac.uk/nag/hsl/. The Voronoi diagram algorithm was coded with Visual C++, version 9.0, using Qt and CGAL tools. In addition, the numerical experiments have been carried out on a PC Intel Celeron with 2.7 GHz CPU and 512MB RAM. The compiler was G++ under Windows operational system.

The test instances have been extracted from a hub location study involving 135 airports in the Brazilian air transportation system, built by Figueiredo *et al.* (2012). Here, we considered a total of 41 airports (nodes) in the air transportation network. The geographic coordinates (latitude and longitude) of each airport was given, and the distances between each pair of airports were computed.

The test instances considered the number of hubs varying from 2 to 6. As suggested by Costa *et al.* (2008), the number of hubs in the Brazilian air transportation network should be from 3 to 6, due to the adequate degree of clustering in the network.

Regarding the solutions obtained by the smoothing hyperbolic approach, which might be outside the network, we applied a heuristic procedure, called Nearest Allocation (NA), implemented in C++ language, to assign the outsider point to the nearest point over the network with lower spoke distances. Figure 1-(a) shows an example of two solution points given by the hyperbolic smoothing technique in cyan color and three points over the network in red color. As we can see, there is an outsider point (a hub location) in the center of the network. Thus, the NA heuristic procedure first allocates the hub location to the nearest point over the network, as we can see in 1-(b). If the current spoke-distances are greater than the spoke-distances associated to the assignment of the hub location to the second nearest network point, then the NA procedure redefine the hub location, resulting in a lower total spoke-distances, as we can see in 1-(c).

Table 1 shows the hub location problem solutions for $p = 2, \ldots, 6$ hubs out of the 41 main Brazilian airports, which are candidates to locate hubs, for the plain hyperbolic smoothing approach (HS) and for the hyperbolic smoothing approach combined with the NA heuristic procedure (HSNA), and for the p-hub median approach.

For the fixed number of hubs $p = 4$, Table 2 shows the results of the plain hyperbolic smoothing (HS), the hyperbolic smoothing combined with the NA procedure (HSNA) and the p-hub median approaches. We notice that the airports set to be hubs are equal for the HSNA approach and for the p-hub median approach.

In Figure 2, the Voronoi diagram refers to the seeds of the 4-hub median approach (in magenta color). In the diagram, the resulting points found by the plain hyperbolic smoothing technique are showed in cyan color, and the spokes are showed in blue color. As we can see, there are only two points with cyan color, because the other two coincided with

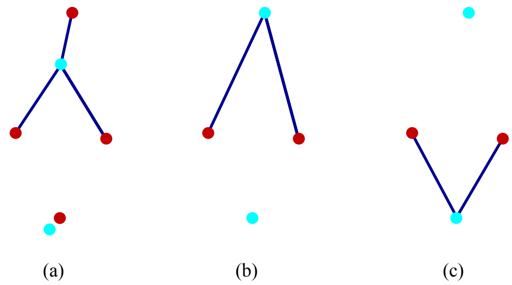

Figure 1: Nearest allocation procedure: (a) initial solution, (b) nearest assignment, and (c) nearest assignment with lower spoke-distances.

Table 1: Results of the hyperbolic smoothing and the p-hub median for $p = 2, \ldots, 6$

	Optimal value		
number of hubs	HS	HSNA	p-hub median
2	300.13	302.413	300.413
3	220.605	223.94	223.94
4	184.992	185.897	185.897
5	158.028	159.592	159.252
6	139.437	140.663	140.663

Table 2: Results for the 4-hubs network design

Approach	Optimal value	Hubs
HS	184.992	not airports
HSNA	185.897	IMP, MAO, MCZ, VCP
p-hub median	185.897	IMP, MAO, MCZ, VCP

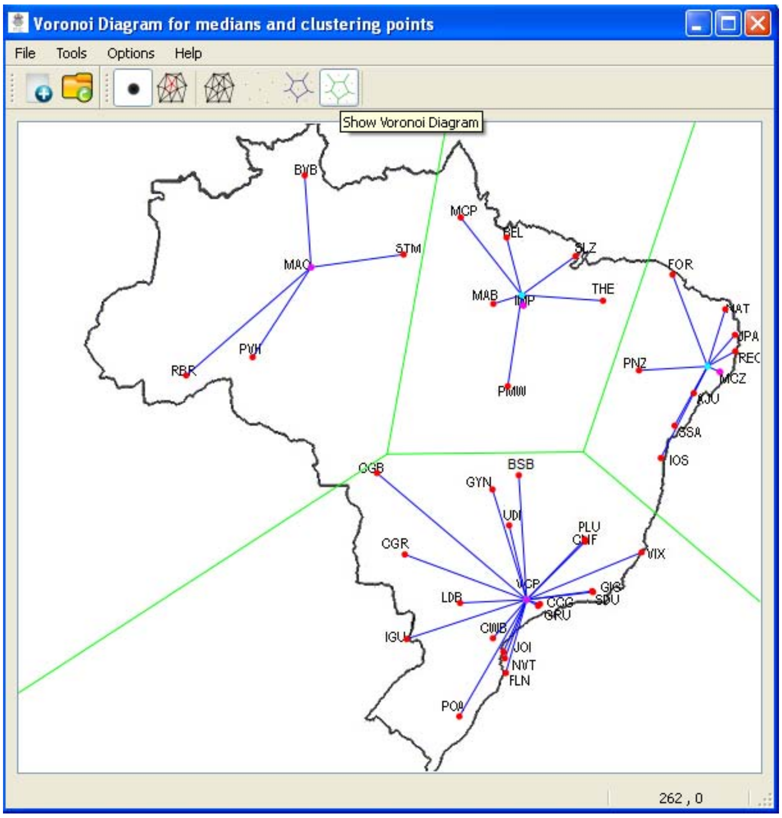

Figure 2: Brazilian map for 4 hubs as seeds of the Voronoi diagram

Table 3: Results for the 6-hubs network design

Approach	Optimal Value	Hubs	Differences
HS	139.437	not airports	VCP not linked to JOI
HSNA	140.663	IMP, MAO, JOI, MCZ, GYN, GIG	VCP linked to JOI
p-median	140.663	IMP, MAO, JOI, MCZ, GYN, GIG	VCP linked to JOI

the medians found, MAO and VCP airports.

For the fixed number of hubs $p = 6$, Table 3 shows the results for the plain hyperbolic smoothing (HS), the hyperbolic smoothing combined with the NA procedure (HSNA) and the p-hub median approaches. We notice that the airports set to be the hubs are equal for the HSNA approach and for the p-hub median approach. Here, there is an example of the benefit of applying the NA procedure. From HS, the VCP airport is linked to a hub point that is not an airport. With the application of the NA procedure, VCP is linked first to GIG hub, but the lower total distances are attained when VCP is linked to JOI hub.

In Figure 3, the Voronoi diagram refers to the seeds of the 6-hub medians approach (in magenta color). In the diagram, the resulting points found by the plain hyperbolic smoothing technique are showed with cyan color, and the spokes are showed in blue color. As we can see, there are only four points with cyan color, because the other two coincided with the medians found, GYN and MAO airports.

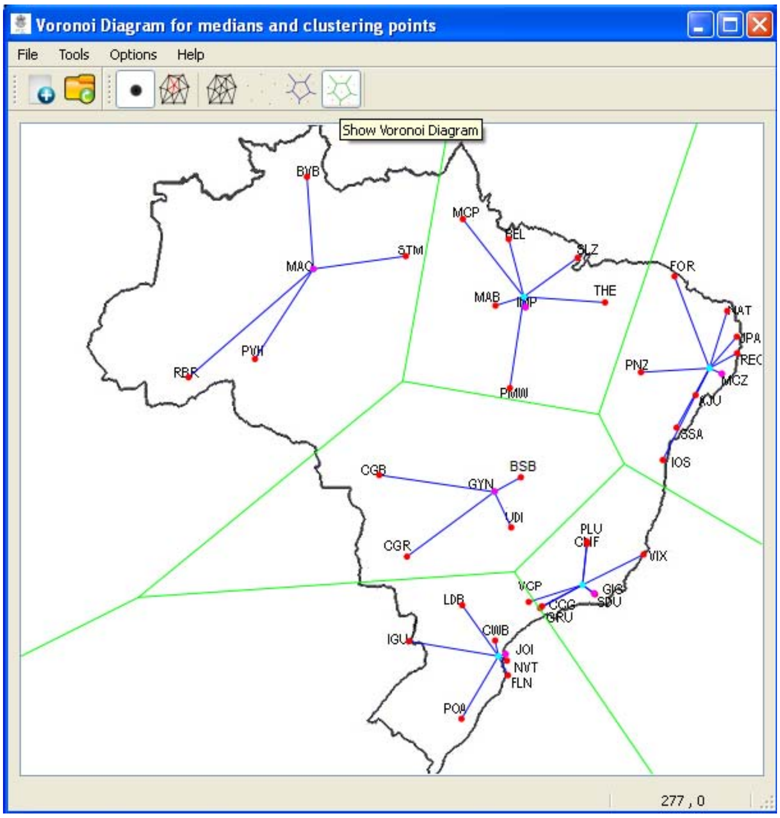

Figure 3: Brazilian map for 6 hubs as seeds of the Voronoi diagram

6 Conclusions

Here, we compared the results of two distinct solutions methods with the support of the Voronoi diagram to design the Brazilian air transportation system by introducing hubs.

Considering instance tests with 41 Brazilian airports and fixed number of hubs varying from 2 to 6, the solutions of the discrete p-hub median approach and the continuous hyperbolic smoothing approach combined with the proposed discretization heuristic were compared in terms of the total spoke-distances. The Voronoi diagram helped to identify the scope of the hubs found by these approaches.

From the computational results, we verified that, with the proposed discretization heuristic, the locations of the hubs are the same considering the number of hubs equal to 4 and 6.

We notice that when the existing airports' infrastructure should be considered, it is more adequate to use a discrete method to determine the locations of the hubs that minimize the total flow costs. On the other hand, when the infrastructure should be developed, it is more adequate to use a continuous method, because from the obtained computational results the optimal total distances would be smaller.

In future work, we plan to consider costs related to the hub locations.

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