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ON PRICING AND COMPOSITION OF MULTIPLE BUNDLES

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Abstract

This paper studies the problem facing a firm of determining the optimal composition and pricing of multiple bundles offered in a market where they compete with other bundles. The analysis assumes that the prices and characteristics of the competitor's bundles are known and that the competition does not react in the short run to the firm's decisions. Consumers are assumed to be rational and to maximize a random utility function. The problem is modeled as a mixed integer non-linear program, which by its nature is difficult to solve using traditional methods. A novel two-phase solution approach is therefore developed. The first phase derives a closed-form expression to solve the optimal pricing subproblem for the bundles assuming their composition is known, and the second phase then uses this expression to arrive at an optimal solution to the composition subproblem.

Keywords: Pricing, Bundling, Dynamic Programming.
Paper Topics: OA; OC; PM.

1. Introduction

Bundling is a key strategy for businesses whose customers have heterogeneous preferences. If price discrimination is not possible, firms must seek other alternatives to maximize benefits such as product/service design or optimal pricing (Voss et al., 2008). In its simplest form, bundling consists in grouping goods and services into a package and selling them at a global price that is generally more attractive than the prices they would sell at if sold separately (Guiltinan and Gordon, 1988).

The practice of bundling is playing an increasingly important role in many industries, and some service sector firms now base their business strategies on this tool (Simon and Wuebker in (Fuerderer et al., 1999)). Examples of its application include holiday packages (return flight, hotel stay and car rental); insurance packages (life and unemployment), restaurant menus (entrée, main dish and dessert) and telecommunications packages (long distance, Internet access and cellular phone).

2. Literature survey

Bundling has been widely investigated from consumer behavior, economic and marketing points of view. As regards consumer behavior, studies such as Yadav and Monroe (1993) and Yadav (1994), focus on the way consumers evaluate different product packages. Suri and Monroe (2000) examine the effects of contextual factors on consumer intentions to buy product packages. The degree to which bundling can stimulate demand

for a specific product promotion has also been analyzed. The results show that promotions on individual elements making up a bundle can significantly reduce consumer evaluations of special offer packages.

In the case of markets where customers have different levels of knowledge of the bundle components, Basu and Vitharana (2009) demonstrate that those with more knowledge show greater variability of reservation price. The authors use an analytic model to determine the conditions for obtaining the maximum benefits with each of three sale strategies: by individual component (no bundling), by bundle only (pure bundling), and mixed.

Certain fundamental results for understanding bundling from an economic and marketing standpoint are given in (Adam and Yellen, 1976). The authors introduce a two-dimensional graphical framework to analyze the effects of bundling as a price discrimination tool. In some specific cases the benefits obtained with price discrimination strategies turn out to be lower than those garnered through the integration of services given that the latter raises very high barriers to entry (Nalebuff, 2004).

Other studies posit numerical criteria for determining which bundling strategy is the most profitable. This line of research analyzes bundles containing unwanted components (which reduce total value) with bundles whose value added is greater than that of the sum of its individual elements (Schamalensee, 1982; Dansby and Conrad, 1984; Schamalensee, 1984). An article by Salinger (1995) extends the results of Adam and Yellen (1976) and Schamalensee (1984), suggesting that with linear demand functions, relatively high costs and positively correlated reservation prices, the incentive to bundle may be strengthened if economies of scope are present.

From a consumer perspective, Hitt and Chen (2005) describe the concept of customized bundling in which customers can choose a bundle composed of M out of a total of N products for a fixed price. They show that this approach outperforms pure bundling and no bundling strategies when there are positive marginal costs and consumer valuations are heterogeneous. More recently, Wu et al. (2008); Yang and Ng (2010) find that customized bundling increases benefits to the firm compared to pure and no bundling approaches when consumers do not value positively all of the component goods.

Other studies suggest an optimization approach that simultaneously decides bundle design and pricing. In Hanson and Martin (1990), a mathematical programming formula is proposed for determining the bundle configuration and price that maximize benefits without explicitly considering the entire range of feasible solutions. In a later article Venkatesh and Mahajan (1993) consider two dimensions of the consumer decision-making process (time and money) in determining the optimal price for a given bundle under different bundling strategies. A pair of publications by Green et al. (Green et al., 1991, 1992) present an algorithm that solves a stochastic mixed integer programming model to determine the price and configuration of a bundle. A mixed integer non-linear programming method has also been devised for studying customized bundles of information goods under different assumptions about company cost structures and consumer valuations (Wu et al., 2008). In a more recent work, Proano et al. (2011) builds a mixed integer non-linear programming model to identify the number of vaccines in bundles of antigens and the range of feasible prices that maximize the sum of producer benefits and consumer surplus.

Finally, the problem of how to determine the composition and price of a single bundle that maximize the total expected benefits to the firm in a competitive market is analyzed by Bitran and Ferrer (2007). This study formulates a mixed integer non-linear programming model and shows that it is possible to identify an optimal

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policy for bundle composition and price.

The present study, which is closely related to the literature just discussed, attempts to determine the optimal composition and price of a set of b bundles a firm intends to offer on the market.

3. Definition of the problem

The problem facing our hypothetical firm is to determine the composition and prices of a set of bundles supplied to a single segment of heterogeneous customers. Thus, the company aims to define a set of b bundles and find the optimal price for each one so as to maximize its benefits. A set of component products or services must be selected to form each bundle, and for each such set there is a group of known choice alternatives. Is important to note that the bundles must differ in composition.

4. Specification of the model

Before specifying the proposed model we set out the necessary notation.

Sets:

$\mathcal{N}$	:	the set of bundles offered on the market by the competition.
$\mathcal{M}$	:	set of components in a bundle. By abuse of notation, we say that $Card(\mathcal{M}) = \mathcal{M}$.
$\mathcal{S}_j$	:	set of alternatives for component j of a bundle.

Indexes:

k, l, t	:	indexes for the bundles offered by the firm.
n	:	index for the bundles offered by the competition.
j	:	index for the set of components of a bundle.
u	:	index for the possible choice alternatives of a component.
i	:	index for all bundles offered on the market.

Parameters:

b	:	number of bundles to be constructed.
$\hat{p}_n$	:	price at which bundle n is offered by the competition.
c_{uj}	:	cost of alternative u of the set of components j.
I_{uj}	:	attractiveness of alternative u of the set of components j.
a_j	:	attractiveness weight of component j in the bundle composition.
γ	:	utility of the bundles offered by the competition and non-purchase.
g	:	number of bundles offered by the competition.
β	:	sensitivity of utility to bundle price.

Variables:

x_{ujk}	:	binary variable indicating whether alternative u of the set of components j is chosen for the composition of bundle k.
X_k	:	representing the composition chosen for the bundle k.
p_k	:	price of bundle k.
q_k	:	probability that a consumer chooses bundle k.
c_{X_k}	:	represents the cost to the firm of the composition chosen for the bundle k.
I_{X_k}	:	represents the attractiveness of the composition chosen for the bundle k.

Decisions that are under the firm's control are the composition of the b bundles it offers to its customer

segment and the prices it sets for the bundles so as to maximize profits. Since we assume that competitors do not react in the short run to the firm's decisions, the model is static rather than dynamic.

Consumers may choose any one, or none, of the bundles offered on the market. They are considered to be random utility maximizers, an assumption common to many consumer choice models. This implies that consumers will prefer the option that gives them the most perceived utility. According to McFadden (1974) they choose over a set of attributes, making an overall evaluation of every possible choice in terms of a random utility function and then picking the one that confers the highest value.

Since the firm offers b bundles to the market, the probability that the firm's bundle i is chosen is given by

$$q(p_i, X_i) = \frac{e^{V(p_i, X_i)}}{\gamma + \sum_{k=1}^{b} e^{V(p_k, X_k)}} \tag{1}$$

where the deterministic utility is $V(p_i, X_i) = I_{X_i} + \beta p_i$ and the attractiveness of the bundle is $I_{X_i} = \sum_{j \in \mathcal{M}} a_j \sum_{u \in \mathcal{S}_j} I_{uj} x_{uji}$, parameter $a_j > 0$ for all $j \in \mathcal{M}$, the parameter $\beta < 0$ is a scalar that expresses the sensitivity of utility to price, and if we let V_0 be the deterministic no-purchase utility $\gamma = e^{V_0} + \sum_{n \in \mathcal{N}} e^{V(p_n, X_n)}$.

Since part of our problem is to simultaneously determine the composition of b bundles, we must prevent the same composition being chosen for two or more of them. This can be ensured by making pairwise comparisons of all the chosen bundles to check that there are no more than $\mathcal{M} - 1$ identical components. Equation (8) below ensures this condition.

With the foregoing as a basis, we now set out our proposed *multiple bundling problem* (MBP) model for determining the composition and pricing of the b bundles.

$$\text{(MBP)} \quad \max_{p_1, \ldots, p_b, X_1, \ldots, X_b} \Pi(p_1, \ldots, p_b, X_1, \ldots, X_b) = \sum_{k=1}^{b} q_k(p_k, X_k)(p_k - c_{X_k}) \tag{2}$$

subject to:

$$q_k(p_k, X_k) = \frac{e^{I_{X_k} + \beta p_k}}{\gamma + \sum_{l=1}^{b} e^{I_{X_l} + \beta p_l}} \quad \forall k = 1, \ldots, b. \tag{3}$$

$$\gamma = e^{V_0} + \sum_{n \in \mathcal{N}} e^{I_{X_n} + \beta \hat{p}_n} \tag{4}$$

$$c_{X_k} = \sum_{j \in \mathcal{M}} \sum_{u \in \mathcal{S}_j} c_{uj} x_{ujk} \quad \forall k = 1, \ldots, b. \tag{5}$$

$$I_{X_k} = \sum_{j \in \mathcal{M}} a_j \sum_{u \in \mathcal{S}_j} I_{uj} x_{ujk} \quad \forall k = 1, \ldots, b. \tag{6}$$

$$\sum_{u \in \mathcal{S}_j} x_{ujk} = 1 \quad \forall j \in \mathcal{M}; \forall k = 1, \ldots, b. \tag{7}$$

$$\sum_{j \in \mathcal{M}} \sum_{u \in \mathcal{S}_j} x_{ujk} x_{ujl} \leq \mathcal{M} - 1 \quad \forall k = 1, \ldots, b; \forall l = k+1, \ldots, b. \tag{8}$$

$$x_{ujk} \in \{0, 1\} \quad \forall j \in \mathcal{M}; \forall u \in \mathcal{S}_j; \forall k = 1, \ldots, b. \tag{9}$$

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$$p_k \geq 0 \qquad \forall \, k = 1, \ldots, b. \tag{10}$$

As can be appreciated, the model is a non-linear programming formulation with continuous and binary variables.

5. Solving the multiple bundling problem

The structure of the MPB was designed to permit the solution process to be divided into two phases, each one solving a single subproblem. In the first phase, the optimal price is determined for each of the b bundles on the assumption that the composition of each of them is known. Then, in the second phase, the optimal prices obtained in the first phase are used in generating the compositions that maximizes benefits to the firm.

5.1. Phase 1: Multiple bundle optimal price subproblem

Given our assumption for this phase that the composition of each bundle is known, the decisions and constraints related to bundle composition are temporarily set aside. Substituting equation (3) into the objective function (2), the unconstrained price optimization subproblem is then just

$$\Pi = \max_{p_1, \ldots, p_b \geq 0} \sum_{k=1}^{b} \frac{e^{I_{X_k} + \beta p_k}}{\gamma + \sum\limits_{l=1}^{b} e^{I_{X_l} + \beta p_l}} (p_k - c_{X_k}). \tag{11}$$

This leads to the following proposition:

Proposition 1 *The optimal price for bundle k is given by the closed form expression*

$$p_k^* = c_{X_k} - \frac{1}{\beta} \left(1 + W \left(\frac{1}{\gamma} \sum_{l=1}^{b} e^{I_{X_l} + \beta c_{X_l} - 1} \right) \right) \qquad \forall \, k = 1, \ldots, b, \tag{12}$$

where $W(\cdot)$ is the Lambert W-function. Note that the optimal price of bundle k depends on the composition of the firm's b bundles. This optimal price is always greater than or equal to the cost to the firm of (c_{X_k}) given that $\beta < 0$ and $W(\cdot) \geq 0$ if its argument is positive, which in this case is true. This implies that prices are always non-negative as well.

5.2. Phase 2: Multiple bundle optimal composition subproblem

Having just obtained a closed-form expression for the optimal prices of the multiple bundles we now address the second subproblem, which is to determine their optimal composition and thus maximize total benefits.

Consider the definition of *Pareto bundles* posited by Bitran and Ferrer (2007). For any two bundles k and l, k dominates l if $I_{X_k} \geq I_{X_l}$ and $c_{X_k} \leq c_{X_l}$. This implies that instead of having to check the entire feasible bundle space Ω, we can confine our search for the solution to the Pareto-efficient bundle frontier Ω^*, which is constructed with the non-dominated bundles.

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It should be noted that for the problem of finding the optimal composition of just one bundle, the frontier is built of points corresponding to individual bundles, but to determine the optimal composition of multiple bundles the points will consist of groups of feasible bundles. However, each of these bundle's contributions to the total objective function can be identified. To this end we state the following proposition:

Proposition 2 *The partial derivatives of the objective function* Π^* *with respect to* I_{X_k} *and* c_{X_k} *are proportional to* q_k^* *for any arbitrary points* I_{X_k} *and* c_{X_k}. *More precisely,*

$$\nabla\Pi^* = \left(\frac{\partial\Pi^*}{\partial I_{X_k}}, \frac{\partial\Pi^*}{\partial c_{X_k}}\right) = \left(\frac{-1}{\beta}, -1\right)q_k^*, \tag{13}$$

where q_k^* *is the probability that bundle* k *is chosen when its price is* p_k^*.

The results in Proposition 2 are of fundamental importance, showing that an increment in the objective function with respect to I_{X_k} is proportional to $-1/\beta$ while an increment with respect to c_{X_k} is proportional to -1. This means that the point on the Pareto-efficient frontier for the multiple bundles problem that maximizes utility is the one whose bundles confer the largest utility increment. The determination of the optimal composition of multiple bundles therefore reduces to identifying the composition of those that maximize the contribution to the objective function. This conclusion is expressed as follows:

$$\max_{X_1, X_2, \ldots, X_b \in \Omega} \sum_{k=1}^{b} \frac{-I_{X_k}}{\beta} - c_{X_k}. \tag{14}$$

One of the bases of our solution approach is that (14) can be separated into b stages, in each of which the optimal composition of only one bundle is determined. This means that in each stage k we must also consider the optimal composition of the bundles constructed in the previous stages ($l = 1, \ldots, k$). For this purpose we will need the following definition:

Definition 1 *(Inner adjacent frontier) Let* Ω_1 *be the set of all feasible bundles and* Ω_1^* *its Pareto-efficient frontier. Select a bundle* $\bar{b}$ *belonging to* Ω_1^* *and eliminate it from the set* Ω_1. *The result is a new set* Ω_2 *of all the feasible bundles, whose Pareto-efficient frontier is denoted* Ω_2^*. *This new construction is called the inner adjacent frontier of* Ω_1^* *under* $\bar{b}$.

The significance of Definition 1 is the linkage it specifies between successive pairs of optimal bundle composition problems. Thus, if we want to obtain the optimal composition of any two bundles in a set of feasible bundles, we look for the first composition (X_1^*) on the Pareto-efficient frontier of the original problem and the second composition on that frontier's inner adjacent frontier under X_1^*.

Having established the foregoing we are now ready to describe our solution approach. In stage 1 we obtain the optimal composition of a single bundle in the space Ω_1 containing all feasible bundles and therefore also on the Pareto-efficient frontier Ω_1^*. Let us call this problem $P(1)$. The solution of $P(1)$ is denoted X_1^*, indicating the composition of the first of the b optimal bundles. The information on this optimal composition X_1^* is passed on from stage 1 to stage 2 to make certain the second bundle is not given the same composition. This is ensured simply by eliminating X_1^* from the feasible bundle space Ω_1, thus obtaining a new Pareto-efficient frontier Ω_2^* which by construction is the inner adjacent frontier of Ω_1^* under X_1^*. Now let us call $P(2)$

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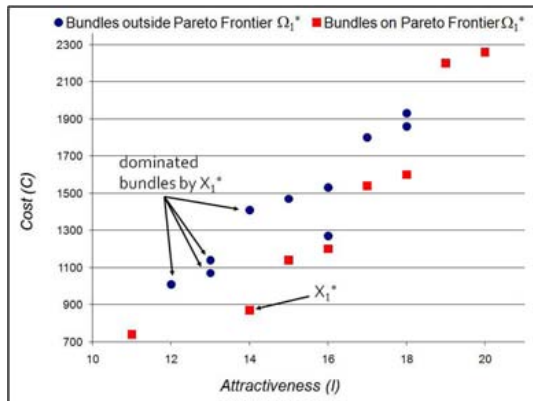Rio de Janeiro, Brazil

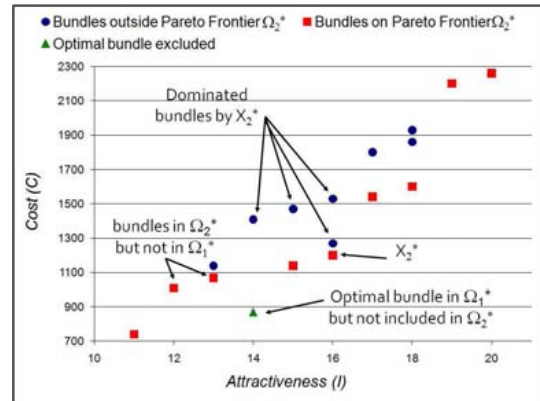Figure 1: Efficient Pareto frontier Ω_1^*.

Figure 2: Efficient Pareto frontier Ω_2^*.

the problem of determining the optimal composition of a single bundle located in the feasible bundle space Ω_2, and therefore also on Ω_2^*. The solution to $P(2)$ will give the optimal composition X_2^*, the composition of the second bundle. This second bundle is eliminated from the feasible bundle space Ω_2, resulting in the construction of a new Pareto-efficient frontier that will be the inner adjacent frontier of Ω_2^* under X_2^*. The optimal composition and the new Pareto-efficient frontier are then passed on to the next stage and the third bundle is determined as before. Repeating this process b times will identify the optimal composition of all b bundles.

The solution approach just outlined is an application of dynamic programming given that it divides the original problem into b sequential subproblems and uses the solution of each of them to obtain the solution to the next one. We can therefore reformulate (14) as a dynamic programming problem expressed as follows:

$$\underbrace{\underset{X_b \in \Omega_b^*}{\text{máx}} \frac{-I_{X_b}}{\beta} - c_{X_b}}_{\text{stage } b} + \left\{ \underbrace{\underset{X_{b-1} \in \Omega_{b-1}^*}{\text{máx}} \frac{-I_{X_{b-1}}}{\beta} - c_{X_{b-1}}}_{\text{stage } b-1} + \ldots + \left\{ \underbrace{\underset{X_1 \in \Omega_1^*}{\text{máx}} \frac{-I_{X_1}}{\beta} - c_{X_1}}_{\text{stage } 1} \right\} \ldots \right\}, \quad (15)$$

where Ω_{k+1}^* is the inner adjacent frontier of Ω_k^* under X_k^* for $k = 1, \ldots, b-1$. We can rewrite (15) as a forward formulation of the Bellman equation (Bellman, 1957) for each stage k as follows:

$$F_k^*(\Omega_k^*) = \underset{X_k \in \Omega_k^*}{\text{máx}} \left\{ \frac{-I_{X_k}}{\beta} - c_{X_k} + F_{k-1}^*(\Omega_{k-1}^*) \right\}. \quad (16)$$

Bellman's optimality principle (Bellman, 1957) ensures that the optimal compositions $X_1^*, X_2^*, \ldots, X_b^*$, each one obtained as the solution of its respective stage-level problem, constitute the optimal solution of the complete subproblem. To solve (16), the Pareto-efficient frontiers must be constructed for each of the stages $k = 1, \ldots, b$. But as has already been observed, their construction is highly complex. However, they can in fact be identified by making use of the inner adjacent dependency existing between the frontiers of two successive stages (k and $k+1$).

Proposition 3 *Let Ω be the feasible bundle space. Also, let Ω_1^* a Pareto-efficient frontier containing d bundles*
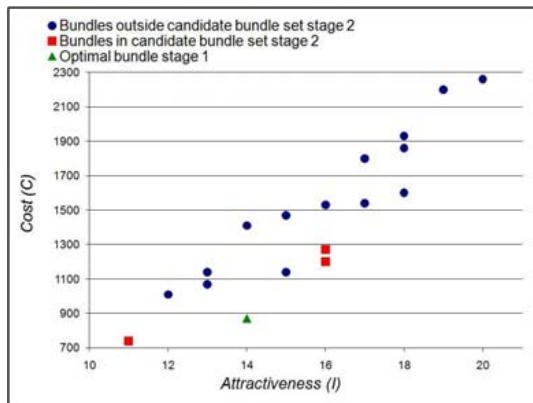
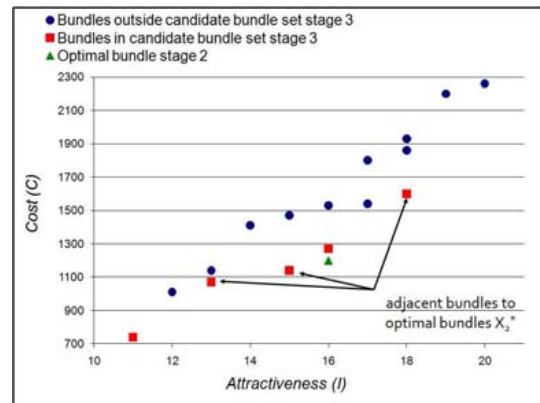Figure 3: Results for stage 1 and sets for stage 2.

Figure 4: Results for stage 2 and sets for stage 3.

whose optimal bundle has the composition X_1^*. Finally, let Ω_2^* be the inner adjacent frontier of Ω_1^* under bundle X_1^*. Then Ω_2^* will contain the $d-1$ bundles other than X_1^* that were on Ω_1^* as well as some of bundles in Ω that were not on Ω_1^* because they were dominated exclusively by X_1^*.

Proposition 3 leads us to conclude that constructing inner adjacent frontiers is also a complex task. Even though $d-1$ bundles on the frontier are known, it is not clear which or how many bundles will appear on a frontier once the bundle that dominated them has been eliminated from the feasible bundle space.

In view of the above, we set out in what follows an alternative procedure for solving (15) without having to build at each stage either the Pareto-efficient or the inner adjacent frontiers. We begin with two useful definitions:

Definition 2 *(Ranked list of a component) If we calculate the contribution of each component alternative to the gradient of the objective function using the expression $f(I, c) = \frac{-I}{\beta} - c$, a descending ordered list of the alternatives can be constructed as a function of the value of $f(I, c)$. This list is called the ranked list of a component.*

Definition 3 *(Adjacent bundle) Let k and l be two bundles. Bundle l is said to be adjacent to bundle k if they are the same in all components except one, for which the chosen alternatives in both cases are in consecutive places on the ranked list. In the special case where the chosen alternative for the component in bundle k is the last one on the list, the chosen alternative for l must be the first on the list (the latter will be used to properly define the algorithm).*

Note that it follows from Definition 3 that a bundle will have as many adjacent bundles as components. This leads to the definition of what is called a candidate bundle set:

Definition 4 *(Candidate bundle set) Let X_1^*, X_2^*,...,X_k^* be the optimal composition of the bundles obtained in stages 1 to k, respectively. Also let $L(X_1^*)$ be the set of bundles adjacent to X_1^*, $L(X_2^*)$ the set of bundles adjacent to X_2^*, and so on up to $L(X_k^*)$, the set of bundles adjacent to X_k^*. Now construct a new set $T(X_{k+1})$ as the union of the k set of adjacent bundles. Extract from $T(X_{k+1})$ those bundles that are repeated and and*

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those bundles whose index value $f(I_X, c_X) = \frac{-I_X}{\beta} - c_X$ is greater than to the optimal composition of the bundle obtained in the last solved stage X_k^*. Finally those bundles whose index is equal to the bundle obtained in the last stage X_k^* remain in the set $T(X_{k+1})$ if not match any of the bundles $X_1^*, X_2^*, \ldots, X_k^*$. We define a set $T(X_{k+1})$ resulting as the set of candidate bundles for determining the optimal composition of the stage k+1 bundle.

With Definitions 2, 3 and 4 we can now develop the following proposition:

Proposition 4 *Let Ω_k be the feasible bundle space in stage k. Also, let Ω_k^* be the Pareto-efficient frontier of Ω_k and X_k^* the optimal composition of bundle k. Then the optimal composition of the stage k+1 bundle with $k = 1, \ldots, b - 1$, denoted X_{k+1}^*, is the bundle in $T(X_{k+1})$ that has the highest index as given by $f(I_X, c_X) = \frac{-I_X}{\beta} - c_X$.*

The importance of Proposition 4 lies in the fact that when we want to solve stage k+1 of (15) we need only to find the optimal composition of bundle k+1 in the set $T(X_{k+1})$ instead of having to determine the Pareto-efficient frontier Ω_{k+1}^*. Note that since the number of adjacent bundles for X_k^* is $\mathcal{M}$, the number of bundles in $T(X_{k+1})$ is bounded above by $k \cdot \mathcal{M}$.

The corollary 3 ensures that multiple bundles have a very similar optimal composition. The change in the candidate bundle set between stage 2 and stage 3 is shown in Figure 4, which also indicates the optimal bundle for the latter stage. Note that the set $T(X_3)$ contains the $T(X_2)$ and $L(X_2^*)$ bundles.

The above-described procedure is captured in the following solution algorithm for the multiple bundle composition problem.

Multiple Bundle Composition Algorithm

Begin
    Do $Rank[j, u] := 0$ for all $j \in \mathcal{M}$ and $u \in \mathcal{S}_j$; (Ranked list building)
    $k := 1$;
    While $k \leq b$ do
      If $k = 1$ then
        Solve stage 1 with algorithm $BF \mapsto X_1^*$
      End If
      If $k > 1$ then
        Determine $L(X_{k-1}^*)$
        Build $T(X_k)$
        Solve stage k:
$$X_k^* := \arg \max_{X_k \in T(X_k)} f(I_{X_k}, c_{X_k}) = \frac{-I_{X_k}}{\beta} - c_{X_k}$$
      End If
      $k = k + 1$
    While End
End

The iterated application of the above procedure ensures that the b bundles will be obtained in exactly b stages. It has the noteworthy advantage of obviating the need to explicitly construct Pareto-efficient frontiers in each stage.

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The proposed algorithm will function correctly as long as all of the component sets $\mathcal{S}_j$ are well-defined in the sense that all of the components have non-negative cost and attractiveness factor values. The computational complexity of the algorithm is of order $O(\mathcal{H}\mathcal{M}+(b-1)\mathcal{M})$, where b is the number of bundles to be composed, $\mathcal{H}$ is the cardinality of the largest set $\mathcal{S}_j$ and $\mathcal{M}$ is the number of components in a bundle.

6. Conclusions and final comments

In this paper has formulated a model non-linear optimization and continuous integer variables giving the optimal price and optimal composition for multiple bundles. However, in general models of this type are extremely difficult to solve. Then we used the structure of the problem to solve in two phases. In the first phase is considered to be known the composition of the bundles of the company, and considering that the demand has been modeled as a multinomial logit, we obtain a unique optimal price for each of the bundles through a closed expression.

In the second phase resolves the problem of optimal composition of these bundles. To do this we replace the closed expression obtained for the optimal price of the bundles in the original formulation. This new model, which is pre-optimized with respect to price, is rewritten as a dynamic programming problem, where in each stage (subproblem) yields the optimal composition of one of the bundles composing. This required recursively build the optimal solution of a stage. This was done by changing from one phase and another Pareto frontier should consider this new stage, since it is the optimal solution (optimal composition of bundles) of the previous stages. This relationship between the Pareto efficient frontier of a stage and the previous stage corresponds to the concept of inner adjacent frontier. Then the problem of determining the optimal composition for the bundles is reduced to a dynamic programming problem, which is a problem every stage of composition of a single bundle over the corresponding Pareto frontier(Bitran and Ferrer, 2007). To solve this problem dynamic programming, we developed a pseudo-polynomial algorithm of order $O(\mathcal{H}\mathcal{M} + (b - 1)\mathcal{M})$. This algorithm solves at each iteration stage dynamic programming problem, and delivers the optimal composition of a new bundle bundles and the list of candidates for the next iteration. Therefore, the algorithm takes as many iterations as desired bundles determined.

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