An $O(n^2)$-Variables Linear Formulation for the Quadratic Assignment Problem

Serigne gueye
Université d'Avignon et des Pays du Vaucluse,
LIA, F-84911 Avignon Cedex 9, France
serigne.gueye@univ-avignon.fr

Philippe Michelon
Université d'Avignon et des Pays du Vaucluse
LIA, F-84911 Avignon Cedex 9, France
philippe.michelon@univ-avignon.fr

Abstract

We present the first linear formulation using distance variables (used previously for the Linear Arrangement Problem) to solve the Quadratic Assignment Problem (QAP). The model involves $O(n^2)$ variables. It has been stengthened by facets and valid inequalities, and numerically tested with QAPLIB instances whose distance matrices are given by the shortest paths in grid graphs. For all the instances, the formulation provides competitive lower bound, in a fewer computational time, in comparison to other litterature techniques. For two of them, our model outperforms the existing techniques both in lower bound quality as well as in CPU time.

Keywords : Quadratic Assignment Problem, Integer Linear Programming.

1 Introduction

The Quadratic Assignment Problem (QAP) has been introduced by Koopmans and Beckmann [20] in 1957. It consists in assigning n entities (plants) to n locations. Locations k and l are separated by a distance of d_{kl}, which might be different from d_{lk}. On the other hand, entities i and j must exchange quantities of a given product f_{ij} and f_{ji} respectively. The cost of assigning i to k is c_{ik} but an assignment also induces a product routing cost which it is assumed to be proportional to the quantities of product to be exchanged and to the distance that separates the entities. In its standard form the mathematical formulation of the (QAP) is based on the binary variables x_{ik}:

$$x_{ik} = \quad = \begin{cases} 1 & \text{if entity } i \text{ is assigned to location } k \\ 0 & \text{otherwise} \end{cases}$$

With these variables, the (QAP) consists in solving:

$$
\begin{array}{lll}
\text{Min} & \sum_{i=1}^{n}\sum_{k=1}^{n} c_{ik}x_{ik} + \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n} f_{ij}d_{kl}x_{ik}x_{jl} & \\
\text{s.t.:} & \sum_{k=1}^{n} x_{ik} = 1 & \forall i \in \{1,\dots,n\} \\
 & \sum_{i=1}^{n} x_{ik} = 1 & \forall k \in \{1,\dots,n\} \\
 & x_{ik} \in \{0,1\} & \forall i,k \in \{1,\dots,n\}
\end{array}
$$

Hence, the (QAP) belongs to the class of 0-1 Quadratic problems. QAP is NP-hard [12], and particularly difficult to solve. The current state-of-the art contains a very large number of contributions that may solve (QAP) optimally or sub-optimally using classical metaheuristic schemes. The exact resolution methods may be classified into two groups. The approaches exploiting, or deriving from, a linear reformulation of the problem, called linearization techniques, and those using a trace reformulation of the problem.

One of the first approaches using the trace formulation of (QAP) is the projection method of Hadley, Rendl and Wolkowitz [16]. Subsequently, this method has been exploited in the Triangular Decomposition method of Karish and Rendl [17] where it is used for computing a lower bound of the remaining part of a decomposition of the quadratic objective function into paths and triangles. One may also consider that the Semi-Definite Programming (SDP) relaxations studied in [18] [28] [29] are methods using the trace formulation of (QAP). The SDP bounds are very competitive. However, due to high computation time requirements, the use of such approaches as basic bounding procedures within branch-and-bound algorithms is up to now not feasible.
The main idea of the linearization techniques is to replace each, or several quadratic terms, by new variables and then to add linear constraints to make the integer linear reformulation equivalent to the original quadratic one. The first implementation of this idea is due to Fortet [9] [10] in the context of pseudo-boolean minimization. It consists in introducing the variable $z_{ikjl} = x_{ik}x_{jl}$ and additional associated constraints making this equality true for any integer solution. These variables have been used by Lawler [21], and Frieze and Yadegar [11]. It has been also used by Adams and Johnson [2] in applying the Reformulation Linearization Technique (RLT) of Adams and Sherali [3] to (QAP). It was shown that the model resulting from the application of the RLT level 1 dominates the Lawler and Frieze-Yadegar ones, as well as the Gilmore-Lawler bound [21][13]. RLT level 1 involves $O(n^4)$ variables. However, its particular structure makes the use of a lagrangean relaxation scheme very efficient. The numerical experiments for RLT show results for nugent instances [23] up to size 20. Following the RLT level 1 algorithm, the level 2 of the same technique has been applied [1]. It implies $O(n^6)$ variables at the benefit of a better lower bound, but at the price of significantly increase the model size that remains difficult to solve for $n > 20$.

Very recently (2011) Fischetti et al [8] have proposed 3 powerful ideas by which 4 previously-unsolved instances (one of them with a size of 128) among the B. Eschermann and H.J. Wunderlich [7] instances have been solved exactly. The first idea deals with symmetry properties deriving from the concept of *clone entities*, that is a property in the flow matrix by which two variables corresponding to clone entities may be swapped without changing the solution cost. The authors then defined clone clusters and used them to reformulate the problem in a rectangular *QAP* where the number of location (say m) is greater than the number of entities (say n). The Kaufman and Broeckx linearization [19] is then applied to this reformulation. Notice that this linearization used some kind of variables also proposed by Glover [15] for non-linear problems. It implies $O(n^2)$

variables, for square QAP, thus $O(n*m)$ in the rectangular case. Finally some decomposition strategies of the flow matrix have been applied for solving the instance of size 128.

The aim of this paper is to propose and study another linear formulation of the (QAP) inducing also $O(n^2)$ additional variables and $O(n^4)$ additional constraints. The formulation is based on distance variables previously used by Caprara and Salazar-Gonzalez [6], as well as Caprara, Letchford and Salazar-González [5] for the Linear Arrangement Problem, a particular case of (QAP). In this latter problem, the distance matrix corresponds to node distances in a simple graph representing a path with unitary edge weight, and the flow matrix is binary.

2 An $O(n^2)$-Variables Linear Formulation

The basic idea of this formulation is to introduce variables D_{ij} representing the distance between entities i and j (which depends on the location of i and j). An alternative would be to introduce variables $F_{k,l}$ representing the quantity of products going from k to l, but we will present here only the formulation with the "distance" variables, the second one being analogous.

With these new variables, the (QAP) can be formulated as the following Mixed-Integer linear Program:

$$(MIP): \text{Min} \quad \sum_{i=1}^{n}\sum_{k=1}^{n} c_{ik}x_{ik} + \sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n} f_{ij}D_{ij} \tag{1}$$

$$\text{s.t.:} \quad \sum_{i=1}^{n} x_{ik} = 1 \qquad \forall i \in \{1,\dots,n\} \tag{2}$$

$$\sum_{k=1}^{n} x_{ik} = 1 \qquad \forall k \in \{1,\dots,n\} \tag{3}$$

$$D_{ij} \geq d_{kl}(x_{ik} + x_{jl} - 1) \qquad \forall i \neq j, k \neq l \in \{1,\dots,n\}, \tag{4}$$

$$x_{ik} \in \{0,1\} \qquad \forall i,k \in \{1,\dots,n\} \tag{5}$$

$$D_{ij} \geq 0 \qquad \forall i,j \in \{1,\dots,n\} \tag{6}$$

In the particular case of the Linear Arrangement Problem in which $d_{kl} = |k-l|$, the formulation corresponds to the one uses in Caprara, Letchford and Salazar-Gonzàlez [5]. Numerical experiments for the Linear Arrangement Problem show a competitive lower bound.

Surprisingly, this linearization has never been used while it may be particulary interesting for (QAP) instances whose distance graphs are given by shortest paths in a grid graph, since those instances have a structure very close to linear arrangement problems. Our contribution is to study this distance variables formulation for general (QAP).

It is easy to check that for any feasible solution, constraints (4) implies that D_{ij} will be greater than the distance between i and j so that, since we are minimizing and $f_{ij} \geq 0$, the above modelisation is valid. Hence, this is a linear model for the Quadratic Assignment Problem with a rather small number of variables. Unfortunately, as observed in Caprara, Letchford and Salazar-González [5], the linear relaxation of this model provides a lower bound of poor quality:

Proposition 1. *If $c_{ik} = 0$ for all i and k, then the linear relaxation of the above problem has an optimal value equal to 0.*

Proof *It is sufficient to observe that* $x_{ik} = \frac{1}{n}$*, for all* i *and* k*, is feasible. Constraints 4 then reduce to* $D_{ij} \geq \frac{2-n}{n} d_{kl}$*, so that* $D_{ij} = 0$ *because of (6).* □

Remark 1. *The Linear Relaxation bound can be slightly improved by substituting* $D_{ij} \geq 0$ *by* $D_{ij} \geq \underline{d}$ *where* $\underline{d}$ *is the smallest distance between two locations.*

It is therefore necessary to strengthen the model. It happens that, besides its low number of variables, its very particular structure makes valid inequalities derivation an easy task.

3 Strengthening the Formulation

As emphasized in proposition 1, the weakness of the model is due to constraints 4 which are not tight enough. In this section, we give one family of valid equalities and two families of facets, linking variables D and x.

Theorem 1. *Let* $d_k = \sum_{l=1}^{n} d_{kl}$*,* $k = 1, 2, \ldots, n$*. The following equalities are valid :*

$$\sum_{j=1}^{n} D_{ij} = \sum_{k=1}^{n} d_k x_{ik} \ , \ i = 1, 2, \ldots, n.$$

Proof: *Let us consider an integer feasible solution of* (MIP)*. It verifies*

$$D_{ij} = \sum_{k=1}^{n} \sum_{l=1}^{n} d_{kl} x_{ik} x_{jl}, \ \forall \ i, j$$

Thus

$$\sum_{j=1}^{n} D_{ij} = \sum_{k=1}^{n} \left[\sum_{l=1}^{n} d_{kl} \sum_{j=1}^{n} x_{jl} \right] x_{ik}$$

It follows with constraint (3) that

$$\sum_{j=1}^{n} D_{ij} = \sum_{k=1}^{n} \left[\sum_{l=1}^{n} d_{kl} \right] x_{ik} = \sum_{k=1}^{n} d_k x_{ik}.$$

□

The remainings two facets are based on the following lemma and some lifting procedures.

Lemma 1. *Let* P *be the convex hull of the integer feasible subdomain of* (MIP) *where* $x_{ik} = 1$ *and* $x_{jl} = 1$*. We have :* $\forall \ i \neq j$*,* $k \neq l$*,* $D_{ij} \geq d_{kl}$ *is a facet of* P*.*

Proof: *First,* $P \subset \mathbb{R}^{(n-2)^2 + n(n-1)}$ *since the number of variables of the subproblem where* $x_{ik} = 1$ *and* $x_{jl} = 1$ *is* $(n-2)^2 + n(n-1)$*.*

Let A *be the matrix corresponding to the assignment constraints (2) and (3). We know, applying polyhedral theory result (see Nemhauser and Wolsey [22]) to our model, that*

$$dim(P) + rank(A) = (n-2)^2 + n(n-1)$$

where $dim(P)$ *stands for the dimension of* P*. This result being in fact valid for any polyhedra and set of equality constraint represented by a matrix* A*. Then* $dim(P) = (n-2)^2 + n(n-1) - rank(A)$*.*

Let F be the set of points of P verifying $D_{ij} = d_{kl}$ and A' the extended matrix containing constraints (2), (3) and $D_{ij} = d_{kl}$. F is a face of P. Since any face is also a polyhedra the previous equality remains valid for F and A'.

$$dim(F) + rank(A') = (n-2)^2 + n(n-1)$$

But $rank(A') = rank(A) + 1$ since D_{ij} are not included in constraints (2) nor in (3). It follows that

$$dim(F) = (n-2)^2 + n(n-1) - rank(A) - 1 = dim(P) - 1$$

Hence, by definition, F is a facet of P. □

With this lemma the following theorem may be introduced.

Theorem 2. *For given k and l, let :*

- $\alpha_{kl} = min\{d_{el} \mid e = 1,...,n; e \neq k\}$
- $\beta_{kl} = min\{d_{km} \mid m = 1,...,n; m \neq l\}$
- $\gamma_{kl} = min\{d_{em} \mid e,m = 1,...,n; e \neq k; m \neq l\}\}$
- $\sigma_{kl}^{(1)} = min\{\beta_{kl} - d_{kl}, \gamma_{kl} - \alpha_{kl}\}$
- $\sigma_{kl}^{(2)} = min\{\alpha_{kl} - d_{kl}, \gamma_{kl} - \beta_{kl}\}$

The following inequalities are facets of the (QAP) feasible set for any i, j, k and l:

1. $D_{ij} \geq \alpha_{kl} + \sigma_{kl}^{(1)} + (d_{kl} - \alpha_{kl})x_{ik} - \sigma_{kl}^{(1)} x_{jl}$
2. $D_{ij} \geq \beta_{kl} + \sigma_{kl}^{(2)} - \sigma_{kl}^{(2)} x_{ik} + (d_{kl} - \beta_{kl})x_{jl}$

***Proof**: We know that $D_{ij} \geq d_{kl}$ is a facet. We can then process, thanks to the structure of the problem, a lifting procedure. Thus, we are looking for M such that:*

$$D_{ij} \geq d_{kl} + M(1 - x_{ik})$$

is valid for any feasible solution verifying $x_{jl} = 1$, that is, we are looking for the optimal value of the problem:

$$Min\{D_{ij} - d_{kl}/constraints\,(2,3,4,5,6), x_{ik} = 0, x_{jl} = 1\}$$

which is equal to $\alpha_{kl} - d_{kl}$ by enumeration on the possible assignments of i.

Hence, $D_{ij} \geq d_{kl} + (\alpha_{kl} - d_{kl})(1 - x_{ik})$ is valid for any feasible solution such that $x_{jl} = 1$, and we are now looking for another M such that

$$D_{ij} \geq d_{kl} + (\alpha_{kl} - d_{kl})(1 - x_{ik}) + M(1 - x_{jl})$$

is valid for any feasible solution. It corresponds to the optimal value of the problem:

$$Min\{D_{ij} - d_{kl} - (\alpha_{kl} - d_{kl})(1 - x_{ik})/constraints\,(2,3,4,5,6), x_{jl} = 0\}$$

In order to solve the problem, its feasible set is partitioned into two subsets, according to the possible values of x_{ik}. If $x_{ik} = 1$, the objective function is $D_{ij} - d_{kl}$ and the value on this particular subset can be found by enumerating the possible assignments for j. Thus, on this subset, the optimal value is $\beta_{kl} - d_{kl}$. On the subset where $x_{ik} = 0$, the objective function is $D_{ij} - d_{kl} - (\alpha_{kl} - d_{kl}) = D_{ij} - \alpha_{kl}$ and the optimal value can be found by enumerating all the possible assignment for i and j, yielding a value of $\gamma_{kl} - \alpha_{kl}$. Thus $M = \sigma^{(1)}_{kl}$, so that the inequality

$$D_{ij} \geq \alpha_{kl} + \sigma^{(1)}_{kl} + (d_{kl} - \alpha_{kl})x_{ik} - \sigma^{(1)}_{kl} x_{jl}$$

is valid and is even a facet since it has been obtained by a lifting process from a facet of the reduced polytope.

The second inequality is symetric and obtained by lifting first $x_{jl} = 1$. □

Remark 2. *It is important to note that the coefficients in these inequalities are independent of i and j.*

The above inequalities have been found by lifting on $x_{ik} = 1$ and $x_{jl} = 1$. However, there exist other possibilities for lifting.

Theorem 3. *For given k, k', l, l', let :*

- $\delta^{(1)}_{l'l} = min\{d_{el'} - d_{el}/e = 1...n\}$
- $\delta^{(2)}_{k'k} = min\{d_{k'm} - d_{km}/m = 1...n\}$

The following inequalities are facets of the (QAP) feasible set for any i, j, k and l:

1. $D_{ij} \geq d_{kl} + \sum_{e=1}^{n}(d_{el} - d_{kl})x_{ie} + \sum_{l'=1}^{n} \delta^{(1)}_{l'l} x_{jl'}$
2. $D_{ij} \geq d_{kl} + \sum_{m=1}^{n}(d_{km} - d_{kl})x_{jm} + \sum_{k'=1}^{n} \delta^{(2)}_{k'k} x_{ik'}$

***Proof**: As in the previous proof, the facets are found by lifting. Let us note that $D_{ij} \geq d_{kl}$ is a facet whenever $x_{ie} = 0, \forall e \neq k$ and $x_{jm} = 0, \forall m \neq l$. We are going to successively lift the inequality with respect to x_{ie}, $e = 1,...,n$. Let us first start with x_{i1}. We are then looking for M such that*

$$D_{ij} \geq d_{kl} + Mx_{i1}$$

is valid for any solution verifying $x_{ie} = 0, \forall e, e \neq k, e \neq 1$ and $x_{jm} = 0, \forall m \neq l$. Therefore M is the optimal value of the problem :

$$\begin{array}{ll} Min & D_{ij} - d_{kl} \\ s\text{-}t: & (2),(3),(4),(5),(6) \\ & x_{jm} = 0, \forall m \neq l, x_{ie} = 0, \forall e \neq k, e \neq 1, x_{i1} = 1 \end{array}$$

Since $x_{jm} = 0, \forall m \neq l \Rightarrow x_{jl} = 1$, *the optimal value of the above problem is* $d_{1l} - d_{kl}$ *and then*

$$D_{ij} \geq d_{kl} + (d_{1l} - d_{kl})x_{i1}$$

is valid for any solution such that $x_{ie} = 0, \forall e, e \neq k, e \neq 1$ *and* $x_{jm} = 0, \forall m \neq l$. *We can then iterate and lift with respect to* x_{i2} *to find an inequality:*

$$D_{ij} \geq d_{kl} + (d_{1l} - d_{kl})x_{i1} + Mx_{i2}$$

with

$$\begin{array}{rl} M = Min & D_{ij} - d_{kl} \\ s\text{-}t: & (2),(3),(4),(5),(6) \\ & x_{jm} = 0, \forall m \neq l, x_{ie} = 0, \forall e \neq k, e \neq 2, x_{i2} = 1 \end{array}$$

Since $x_{i2} = 1 \Rightarrow x_{ie} = 0 \forall e \neq 2$, *the expression is the same as in the lifting process of* x_{i1} *and then*

$$D_{ij} \geq d_{kl} + (d_{1l} - d_{kl})x_{i1} + (d_{2l} - d_{kl})x_{i2}$$

is valid for any solution such that $x_{ie} = 0, \forall e, e \neq k, e \geq 3$ *and* $x_{jm} = 0, \forall m \neq l$.

By iterating this process, it happens that:

$$D_{ij} \geq d_{kl} + \sum_{e=1}^{n} (d_{el} - d_{kl})x_{ie}$$

is valid for any solution verifying $x_{jm} = 0, \forall m \neq l$. *We now have to lift with respect to* j, *that is we first look for* M *such that*

$$D_{ij} \geq d_{kl} + \sum_{e=1}^{n} (d_{el} - d_{kl})x_{ie} + Mx_{j1}$$

is valid for$x_{jm} = 0, \forall m \neq l, m \geq 2$. *Hence*

$$\begin{array}{rl} M = Min & D_{ij} - d_{kl} - \sum_{e=1}^{n}(d_{el} - d_{kl})x_{ie} \\ s\text{-}t: & (2),(3),(4),(5),(6) \\ & x_{jm} = 0, \forall m \neq l, m \geq 2, x_{j1} = 1 \end{array}$$

This problem can be solved by enumeration of the possible assignments of i. *If* i *is assigned to* k', *then the expression of the objective function reduces to* $d_{k'1} - d_{kl} - d_{k'l} + d_{kl} = d_{k'1} - d_{k'l}$, *so that* $M = \delta^{(1)}_{1l}$. *once again, the process can be iterated to find out the first inequality of the theorem. The second is symetric and has been obtained bu lifting firs the variables related to* j.□

4 Application to Grid Instances of the QAPLIB

To evaluate the quality of the model, some preliminary numerical tests have been performed on some QAPLIB instances [4]. We are particularly interested in problems for which the distance matrix is given by shortest paths in a grid graph because of the numerical challenge that they represent. Their particular structure also allows to generate other valid inequalities presented below (theorems 5 and 6). The corresponding problems are the instances of Nugent et al [23], of Scriabin et Vergin [24], Skorin-Kapov [25], Thonemann et Bölte [26], and of Wilhelm et Ward [27].

Actually no instance of Skorin-Kapov [25] and Wilhelm et Ward [27], had been solved optimally. Only one instance among the 3 of Thonemann et Bölte [26] had been solved exactly. For all unsolved problems an upper bound is known as well as the relative gap between the best lower bound of the litterature. During many years the best lower bounds was provided by the Triangular Decomposition Technique (TDM) of Karish et Rendl [17]. The recent applications of *RLT* (Reformulation-Linearization-Technique) have significanlty improved these bounds. In this paper, we present a comparison of the various existing bounds to the lower bound derived from our formulation, strengthened by the following additional valid inequalities.

Theorem 4. *Let i, j, h verify $1 \leq i < j < h \leq n$. The following triangular inequalities are valid.*

$$\begin{array}{l} D_{ij} \leq D_{ih} + D_{jh} \\ D_{ih} \leq D_{ij} + D_{jh} \\ D_{jh} \leq D_{ij} + D_{ih} \end{array}$$

Proof: *Notice first that in the case of grid graphs the distance matrix $d = \{d_{kl}\}$ is symetric. Hence we just need the upper triangular part of the matrix variable D. This explains the inequalities $1 \leq i < j < h \leq n$.*

Since d derives from a grid graph, it is a metric. Thus D must verify the triangular inequalities of any metric. □

Moreover the grid structure allows to add the following valid inequalities.

Theorem 5. *Let i, j, h verifying $1 \leq i < j < h \leq m$. The following inequalities are valid :*

$$D_{ij} + D_{ih} + D_{jh} \geq 4$$

Proof: *Suppose, by contradiction, that $D_{ij} + D_{ih} + D_{jh} \leq 3$ then, as $D_{ij} \geq 1$, we have i, j $d_{ij} = d_{ih} = d_{jh} = 1$. It follows that the three entities are located in nodes of a cycle of size 3. This is impossible since a grid graph do not contain any cycle of size 3.* □

Notice that these inequalities correspond to a particular case of clique inequalities proposed for the Linear Arrangement Problem (see Caprara et al [5]). Now, by considering four indices instead of three, a similar valid inequality can be derived.

Theorem 6. *Let i, j, h, r verifying $1 \leq i < j < h < r \leq m$. The following inequalities are valid :*

$$D_{ij} + D_{ih} + D_{ir} + D_{jh} + D_{jr} + D_{hr} \geq 8$$

We report in table 1 numerical results obtained by solving the linear relaxation of (MIP) strengthened by the valid inequalities above (the facets of the previous section have not been incorprated in our model yet). Ilog Cplex 12.2 has been used. In the table,"*Prob*" denotes the instance name, "*m*" the number of nodes of the grid, "*Dens*" the density of the flow matrix, $V(\overline{MIP})$ the optimal value of the (MIP) linear relaxation, *GLB* the Gilmore et Lawler [14] [21] bound, "*PB*" the projection method bound [16], "TD" the triangular decomposition bound, "RLT_1" (resp. "RLT_2") the Reformulation Linearization Technique level 1 (resp. 2) bound, UB the best upper bound of the litterature, and $CPU(sec.)$ the computational time of $V(\overline{MIP})$. The articles from which these bounds have been extracted do not necessarily contain experiments for all the instances considered here. Each time the experiment has not been conducted we indicate it by "$-$". ∞ stands for the impossibility to solve the problem because of unsufficient computer memory.

For the instances *nug*12, *nug*15 and *nug*20, our approach provides the third lower bound, behind RLT_2, and the Triangular Decomposition method. Notice that for the instance *nug*20, RLT_2

furnishes the best value (2508) around 7 hours. In comparison, our model provide in 0.8 sec. a worse bound but which is still very close to the best upper bound.

For Skorin-Kapov [25] ($sko*$) instances, until $sko56$ our bound is in second position, and for larger instances in third. The running times of the other methods are only known for "TD": $sko56$ (600 $sec.$), $sko72$ (1800 $sec.$) and $sko90$ (2400 $sec.$). However, in this comparison, we must take into account the fact that they had been obtained on an old computing environment, a Personal Computer 486 at 66 Mhz, thus with a performance much lower than our current one, a Dell laptop with 3456 MBits RAM at 2.40 GHz. For the instances $scr20$, $ste36a$ our bound is better both in values and computational times.

5 Conclusion

We have presented and tested the first formulation using distance variables for general (QAP) problems. The original linear formulation, has a poor bound. It has been improved by some facets linking distance variables D and assignment ones x. The model has been applied in the particular case of *QAP* problems on grid graphs. In these cases, the model has been strengthened with valid inequalities that take into account the metric property of the distance variables, as well as the grid structure. The numerical experiments to evaluate the quality of the lower bound show that the model is very competitive. It allows to compute lower bounds very close to the best known upper bounds in a reduced time. The quadratic number of variables makes possible to solve large size instances at the opposed, for instance, of the *RLT* techniques which cannot solve dimension greater than 20 without considerably increasing the computation time.

The future research directions will deal with solving exactly the grid instances with a Branch-and-Cut framework using our lower bound. We also aim before to enforce the formulation by extending the lifting procedures and valid inequalities of theorem 5 and 6, and taking advantage on similar studies for the Linear Arrangement Problem.

Prob	*m*	*Dens*(%)	$V(\overline{MIP})$	*GLB*	*PB*	*TD*	RLT_1	RLT_2	*UB*	*CPU*(*sec.*)
*nug*12	12	68	540.3	493	–	–	523	578	578.0	0.1
*nug*15	15	71	1083.1	963	–	1083	1041	1150	1150.0	0.1
*nug*16*b*	16	70	1153.8	–	–	–	–	–	1240.0	0.2
*nug*20	20	74	2387.6	2057	2196	2394	2182	2508	2570.0	0.8
*nug*25	25	66	3475.0	–	–	–	–	–	3744.0	2.3
*nug*30	30	67	5687.4	4539	5266	5772	–	–	6124.0	4.8
*sko*42	42	70	14592.9	11311	13830	14934	–	–	15812.0	29.8
*sko*49	49	68	21145.6	16161	20715	22004	–	–	23386.0	47.7
*sko*56	56	68	30882.7	23321	30701	32610	–	–	34458.0	95.5
*sko*64	64	68	42770.6	32522	43890	45536	–	–	48498.0	203.7
*sko*72	72	69	58194.7	44280	60402	62691	–	–	66256.0	414.6
*sko*81	81	70	79362.3	60283	82277	86072	–	–	90998.0	1142.6
*sko*90	90	69	100068.6	75531	105983	108493	–	–	115534.0	2233.6
*sko*100*a*	100	69	130662.4	98953	139365	142668	–	–	152002.0	3695.2
*sko*100*b*	100	68	131767.3	99028	141251	143872	–	–	153890.0	4540.1
*sko*100*c*	100	68	126655.6	95979	135011	139402	–	–	147862.0	3922.7
*sko*100*d*	100	68	127248.7	95921	136979	139898	–	–	149576.0	3837.7
*sko*100*e*	100	68	127574.9	95551	136996	140105	–	–	149150.0	3870.7
*sko*100*f*	100	68	127186.1	96016	136860	139452	–	–	149036.0	3968.8
*scr*12	12	42	30334.3	–	–	–	–	–	31410.0	0.1
*scr*20	20	32	96018.0	86766	16113	87968	–	–	110030	0.4
*ste*36*a*	36	27	8243.1	7124	−11770	6997	–	–	9526	10.6
*tho*30	30	49	136296.4	90578	119255	136447	–	–	149936.0	4.9
*tho*40	40	40	205950.0	143804	191042	214218	–	–	240516.0	14.7
*tho*150	150		∞	4123652	7350920	7620628	–	–	8133398	∞
*wil*50	50	89	44784.4	38069	45731	47098	–	–	48816.0	38.8
*wil*100	100	90	242973.6	210299	260827	263909	–	–	273038.0	2868.9

Table 1: Lower bounds

References

[1] W.P. Adams, M. Guignard, P.M. Hahn, and W.L. Hightower. A level-2 reformulation-linearization-technique bound for the quadratic assignment problem. *Discrete Optimization*, 180:983–996, 2007.

[2] W.P. Adams and T.A. Johnson. Improved linear programming-based lower bounds for the quadratic assignment problem. *DIMACS Series in Discrete Mathematics and Theoretical Computer Science*, 16:43–75, 1994.

[3] W.P. Adams and H.D. Sherali. Linearization strategies for a class of zero-one mixed integer programming problems. *Operations Research*, 38(2):217–226, 1990.

[4] R. E. Burkard, S. E. Karisch, and F. Rendl. Qaplib - a quadratic assignment problem library. *Journal of Global Optimization*, 10:391–403, 1997.

[5] A. Caprara, A.N. Letchford, and J.J. Salazar-Gonzàlez. Decorous lower bounds for minimum linear arrangement. *INFORMS Journal on Computing*, 23(1):26–40, 2011.

[6] A. Caprara and J.J. Salazar-Gonzàlez. Laying out sparse graphs with provably minimum a bandwidth. *INFORMS Journal on Computing*, 17:356–373, 2005.

[7] B. Eschermann and H.J. Wunderlich. Optimized synthesis of self-testable finite state machines. In *20th International Symposium on Fault-Tolerant Computing (FFTCS 20), Newcastle upon Tyne*, 26-28th June, 1990.

[8] M. Fischetti, M. Monaci, and D. Salvagnin. Three ideas for the quadratic assignment problem. *Operations Research (to appear)*, 2011.

[9] R. Fortet. L'algèbre de boole et ses applications en recherche opérationnelle. *Cahier du Centre d'Etudes de Recherche Opérationnelle*, 1:5–36, 1959.

[10] R. Fortet. Application de l'algèbre de boole en recherche opérationnelle. *Revue Française de Recherche Opérationnelle*, 4:17–26, 1960.

[11] A.M. Frieze and J. Yadegar. On the quadratic assignment problem. *Discrete Applied Mathematics*, 5:89–98, 1983.

[12] M. Garey and D. Johnson. *Computers and Intractibility: A Guide to the Theory of NP-Completeness*. W.H. Freeman & Company, 1979.

[13] P.C. Gilmore. Optimal and suboptimal algorithms for the quadratic assignment problem. *SIAM Journal on Applied Mathematics*, 10:305–313, 1962.

[14] P.C. Gilmore. Optimal and suboptimal algorithms for the quadratic assignment problem. *Journal of the Society for Industrial and Applied Mathematics*, 10(2):305–313, 1962.

[15] F. Glover. Improved linear integer programming formulations of nonlinear integer problems. *Management Science*, 22(4):455–460, 1975.

[16] S.W. Hadley, E. Rendl, and H. Wolkowicz. A new lower bound via projection for the quadratic assignment problem. *Mathematics of Operations Research*, 17:727–739, 1992.

[17] S.E. Karisch and F. Rendl. Lower bounds for the quadratic assignment problem via triangle decompositions. *Mathematical Programming*, 71:137–151, 1995.

[18] S.E. Karish. *Nonlinear Approaches for Quadratic Assignment and Graph Partition Problems*. Ph.d. thesis, Technical University Graz, Austria, 1995.

[19] L. Kaufman and F. Broeckx. An algorithm for the quadratic assignment problem using benders' decomposition. *European Journal of Operational Research*, 2:204–211, 1978.

[20] T. C. Koopmans and M. J. Beckmann. Assignment problems and the location of economic activities. *Econometrica*, 25:53–76, 1957.

[21] E.L. Lawler. The quadratic assignment problem. *Management Science*, 9(4):586–599, 1963.

[22] G.L. Nemhauser and L.A. Wolsey. *Integer and combinatorial optimization*. Wiley, 1988.

[23] C.E. Nugent, T.E. Vollman, and J. Ruml. An experimental comparison of techniques for the assignment of facilities to locations. *Operations Research*, 16(1):150–173, 1968.

[24] M. Scriabin and R.C. Vergin. Comparison of computer algorithms and visual based methods for plant layout. *Management Science*, 22:172–187, 1975.

[25] J. Skorin-Kapov. Tabu search applied to the quadratic assingnment problem. *ORSA Journal on Computing*, 2(1):33–45, 1990.

[26] U.W. Thonemann and A. Bölte. An improved simulated annealing algorithm for the quadratic assignment problem. Working paper, School of Business, Department of Production and Operations Research, University of Paderborn, Germany, 1994.

[27] M.R. Wilhem and T.L. Ward. Solving quadratic assignment problems by simulated annealing. *IIE Transaction*, 19(1):107–119, 1987.

[28] Q. Zhao. *Semidefinite Programming for Assignment and Partitioning Problems*. Ph.d. thesis, University of Waterloo, Ontario, Canada, 1996.

[29] Q. Zhao, S. E. Karisch, F. Rendl, and H. Wolkowicz. Semidefinite relaxations for the quadratic assignment problem. *Journal of Combinatorial Optimization*, 2:71–109, 1998.