IMPROVED LAGRANGIAN BOUNDS AND HEURISTICS FOR THE GENERALIZED ASSIGNMENT PROBLEM

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ABSTRACT

Modified lagrangian bounds are proposed for the generalized assignment problem and combined with greedy heuristics to get lagrangian based feasible solutions. Numerical results for problem instances with number of agents close to number of tasks are provided.

KEY WORDS: Generalized assignment problem, Lagrangian relaxations, greedy Heuristics

OC –Combinatorial Optimization, MH - Metaheuristics

1. Introduction

Most large scale optimization problems exhibit a structure that can be exploited to construct efficient solution techniques. In one of the most general and common forms of the structure the constraints set of the problem can be divided into "easy" and "complicating". In other words, the problem would be an "easy" problem if the complicating constraints could be removed. One typical example is a block-separable problem decomposing into a number of smaller independent subproblems if the binding constraints could be relaxed, see Lasdon (2002).

A well-known way to exploit this structure is to form the Lagrangian relaxation with respect to complicating constraints, see Lasdon (2002), Lemaréchal (2001), and Lemaréchal (2007). That is, the complicating constraints are relaxed and a penalty term is added to the objective function to discourage their violation. The optimal value of the Lagrangian problem, considered for fixed multipliers, provides a lower bound (for minimization problem) for the original optimal objective. The problem of finding the best, i.e. bound minimizing Lagrangian multipliers, is called the Lagrangian dual. Lagrangian bounds are widely used as a core of many numerical techniques, e.g. in branch-and-bounds schemes for integer and combinatorial problems. Lagrangian solution is also used as a starting or reference point for heuristic techniques, see Boschetti and Maniezzo (2009).

There are often different ways in which a given problem can be relaxed in a Lagrangian fashion. Suppose, for example, that the set of original constraints can be divided into two subsets and, when considered separately, both have "easy" structures. That is, dualizing either the first subset of constraints, or the second, we get two attractive while different Lagrangian relaxations. Such a structure can be found in the generalized assignment problem, the multiple knapsack problem, the facility location problem, to mention a few. In what follows we will refer to this property as a double decomposable structure.

In this paper we apply to the generalized assignment problem the approach to tighten the Lagrangian bounds proposed in Litvinchev (2007), Litvinchev (2010), for problems with double decomposable structure. The approach can be interpreted in two ways. First, it can be seen as a reformulation of the original problem aimed to split the resulting Lagrangian problem into two subproblems. Second, it can be interpreted as a search for tighter estimation of the penalty term arising in the Lagrangian problem. Lagrangian based greedy techniques are used to get feasible solutions to the problem. Numerical results are presented to demonstrate the quality of primal and dual bounds.

2. Deriving the modified bound

The generalized assignment problem (GAP) is a well known NP-hard combinatorial optimization problem, see Burkard (2009), Pentico (2007). It considers a situation in which n jobs have to be processed by m agents. The agents have capacities expressed in terms of a resource which is consumed by job processing. The objective is to minimize cost of assignment the jobs to agents such that each job is assigned to exactly one agent subject to the agents available capacity.

Let $I = \{1, \dots, m\}$ be the set of agents and $J = \{1, \dots, n\}$ the set of the jobs. A standard integer programming formulation for the GAP is the following:

$$z_{ip} = \quad min \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} \qquad (1)$$

$$s.a: \qquad \sum_{i=1}^{m} x_{ij} = 1, \qquad \forall j \in J \qquad (2)$$

$$\sum_{j=1}^{n} a_{ij} x_{ij} \leq b_i, \qquad \forall i \in I \qquad (3)$$

$$x_{ij} \in \{0,1\}, \qquad \forall i \in I, j \in J \qquad (4).$$

where c_{ij} is the assignment cost of job j to agent i, a_{ij} the resource required for processing job j by agent i, and b_i is the available capacity of agent i. Decision variables x_{ij} are set to 1 if job j is assigned to agent i, 0 otherwise. Constraints (2) together with the integrality conditions on the variables, state that each job is assigned to exactly one agent. Constraints (3) insure that the resources of the agents are not exceeded.

The problem (1)-(4) has a double decomposable structure. Relaxing assignment constraints (2) results in independent subproblems for each agent i, while relaxing resource constraints (3) gives independent subproblems for each job j.

To use the double decomposable structure of GAP consider the modified problem:

$$z_M = \qquad min \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} \qquad (5)$$

$$s.a: \qquad \sum_{i=1}^{m} x_{ij} = 1, \qquad \forall j \in J \qquad (6)$$

$$\sum_{j=1}^{n} a_{ij} x_{ij} \leq \sum_{j=1}^{n} a_{ij} y_{ij}, \qquad \forall i \in I \qquad (7)$$

$$\sum_{j=1}^{n} a_{ij} y_{ij} \leq b_i, \qquad \forall i \in I \qquad (8)$$

$$\sum_{i=1}^{m} y_{ij} = 1, \qquad \forall j \in J \qquad (9)$$

$$\sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} y_{ij} \qquad (10)$$

$$x_{ij}, y_{ij}, \in \{0,1\}, \qquad \forall i \in I, j \in J \qquad (11).$$

A feasible solution x to (1)-(4) is feasible to (5)-(11) with $y = x$, while x-part of a feasible solution (x, y) to (5)-(11) is feasible to (1)-(4) by (7) and (8). Hence $z_{ip} = z_M$. Dualizing constraints (7), (9) and (10) with multipliers, $v \geq 0, u, \pi$ respectively we get the Lagrangian bound:

$$z_{ip} = z_M \geq L_M(v, u, \pi) =$$

$$= min\{(1-\pi) \sum_{i,j} c_{ij} x_{ij} + \pi \sum_{i,j} c_{ij} y_{ij} + \sum_i v_i \left(\sum_j a_{ij} x_{ij} - \sum_j a_{ij} y_{ij}\right) + \sum_j u_j \left(1 - \sum_i y_{ij}\right)\} \quad (12)$$

$$\sum_{i=1}^{m} x_{ij} = 1, \forall j \in J \qquad (13)$$

$$\sum_{j=1}^{n} a_{ij} y_{ij} \leq b_i, \forall i \in I \qquad (14)$$

$$x_{ij}, y_{ij}, \in \{0,1\}, \qquad \forall i \in I, j \in J.$$

Rearranging the terms in the function objective (12) we get:

$$z_{ip} = z_M \geq L_M(v,u,\pi) = \min\left\{\sum_{i,j}\left[(1-\pi)c_{ij} + v_i a_{ij}\right]x_{ij} + \sum_{i,j}\left[\pi c_{ij} - v_i a_{ij} - u_j\right]y_{ij} + \sum_j u_j\right\} \quad (15)$$

$$\sum_{i=1}^{m} x_{ij} = 1, \forall j \in J \quad (16)$$

$$\sum_{j=1}^{n} a_{ij}y_{ij} \leq b_i, \forall i \in I \quad (17)$$

$$x_{ij}, y_{ij}, \in \{0,1\}, \forall i \in I, j \in J.$$

The Lagrangian problem (15)-(17) can be reduced to two independent problems in x and in y. Moreover, the problem in x decomposes into J independent subroblems, while the problem in y decomposes into I independent subproblems. Note also that the problem in x has constraints $\sum_i x_{ij} = 1, \forall j \in J$ which are totally unimodular, so we can relax integrality conditions on x without loosing the optimal solution.

Denote:

$$X^j = \left\{\sum_i x_{ij} = 1, x_{ij} \geq 0\right\}$$

$$Y^i = \left\{\sum_j a_{ij}y_{ij} \leq b_i, y_{ij} \in \{0,1\}\right\}$$

Then we get

$$z_{ip} \geq \varphi(\pi,u,v) \text{for all } v \geq 0, u, \pi \quad (18)$$

where

$$\varphi(\pi,u,v) = \eta(\pi,u) + \xi(\pi,u,v) + \sum_j u_j$$

$$\eta(\pi,u) = \sum_j min_{x\in X^j}\left\{\sum_i\left[(1-\pi)c_{ij} + v_i a_{ij}\right]x_{ij}\right\}$$

$$\xi(\pi,u,v) = \sum_i min_{y\in Y^i}\left\{\sum_j\left[\pi c_{ij} - v_i a_{ij} - u_j\right]y_{ij}\right\}$$

The corresponding dual bound be

$$w_{MD} = max_{u,v\geq 0,\pi}\varphi(\pi,u,v) \quad (19)$$

It was shown in Litvinchev et al. (2010) that in (19) is at least as good as any of two standard Lagrangian bounds obtained by dualizing either constraints (2) or (3).

The value of the modified dual bound (19) can be calculated using constraints generation scheme (Benders technique), see Lasdon (2012) and Martin (1999).

3. Getting feasible solutions

Various heuristic approaches were proposed for GAP to get feasible solutions (see, e.g, Jeet (2007), Laguna et al. (1995), Romeijn (2000), Yaguira (2007) and others the references trerein). We used a simple greedy approach similar to Romeijn (2000) and adjusted in a Lagrangian manner. The method starts from an empty assignment and then proceeds by incorporating the most promising element according to a certain evaluation function. This function reflects the intention to get a feasible solution "close" to the Lagrangian one. The generic algorithm can be presented as follows:

Step 0: Set $x_{ij}^g = 0, \forall\, i,j$. Set $J^0 = J$ and $\beta_i = b_i, \forall\, i$.
Step 1: Let $I_j = \{i \mid a_{ij} \le \beta_i\}$, $\forall j$. If $I_j = \emptyset$ for some j, STOP, The method can´t construct a feasible solution.
Step 2: Set $i_j = \arg\text{mín}\{f(i,j) \mid i \in I_j\}$, $\forall j \in J^0$, and $\rho_j = \text{mín}\{f(s,j) - f(i_j,j) \mid s \in I_j,\ s \neq i_j\}$, $\forall j \in J^0$.
Step 3: Let $j^* = \arg\text{máx}\{\rho_j \mid j \in J^0\}$ and set: $x_{i_{j^*}j^*}^g = 1$, $\beta_{i_{j^*}} \leftarrow \beta_{i_{j^*}} - a_{i_{j^*}j^*}$, $J^0 = J^0 - \{j^*\}$.
Step 4: If $J^0 = \emptyset$, STOP, x^g being a feasible solution. Otherwise, return to Step 1.

Eight evaluation functions $f(i,j)$ were used in Step 2. The first four functions are as follows:

$$f^1(i,j) = (\alpha - t_{ij})c_{ij}, \qquad f^2(i,j) = (\alpha - t_{ij})a_{ij},$$

$$f^3(i,j) = (\alpha - t_{ij})\frac{a_{ij}}{b_i}, \qquad f^4(i,j) = (t_{ij} - \alpha)\frac{c_{ij}}{a_{ij}}$$

where t_{ij} be a Lagrangian solution to problem (12). The next four functions $f^5(i,j), \dots, f^8(i,j)$ defined similarly for t_{ij} be a Lagrangian solution to problem (13).

The use of the factor $(\alpha - t_{ij})$, for $\alpha > 1$, is intended to force the greedy technique to select variables with larger values of Lagrangian solutions. We used $\alpha = 1.5$.

4. Numerical Results

To test the quality of primal and dual bounds, we used three types of benchmark GAP instances, see Chu and Beasley (1997) and Laguna et. Al. (1995) called types C, D, and E, generally in increasing order of difficulty:

- Type C: a_{ij}, are random integers from U[5,25], c_{ij} are random integers from U[10,50] and b_i are 0.8 $\sum_j \frac{a_{ij}}{m}$.
- Type D: a_{ij}, are random integers from U[1,100], $c_{ij} = 111 - a_{ij} + e_1$ where e_1 is a random integers from U[−10,10], and b_i are 0.8 $\sum_j \frac{a_{ij}}{m}$.
- Type E: $a_{ij} = 1 - 10 \ln e_2$, where e_2 is a random number from U[0,1], $c_{ij} = {}^{1000}/_{a_{ij}} - 10e_3$ where e_3 is a random integers from U[0,1], and b_i are 0.8 $\sum_j \frac{a_{ij}}{m}$.

We focused on instances with m relatively close to n which are known to be hard to solve Martello and Toth (1990) since the linear relaxation provides a poor lower bound. For each combination of $m \times n$, 20 instances were generated for each of three classes C, D, and E. The average results are presented in Tables 1-3 for the following indicators:

$$gap_{dual} = \frac{z_{ip} - w_{MD}}{z_{ip}} 100\% \qquad gap_{lp} = \frac{z_{ip} - z_{lp}}{z_{ip}} 100\% \qquad gap_{prim} = \frac{z_{fact} - z_{ip}}{z_{ip}} 100\%$$

where z_{lp} is the lower bound obtained by the linear programming relaxation and z_{fact} is the original objective function value corresponding the best feasible solution found.

The modified dual problem was solved by Benders constraint generation scheme. In each iteration of this technique eight greedy methods were applied simultaneously to the current

Lagrangian solution (x, y) and the best feasible solution was stored. The last two columns in the Tables indicate how frequently (in %) a feasible solution was improved by greedy techniques with evaluation functions f^1 and f^2. We present these indicators only for these two functions since they were the most frequently used.

Table 1. Results for class C

$m \times n$	gap_{lp}	gap_{dual}	gap_{prim}	f^1	f^5
8×10	13.87	0.53	0.00	48	23
18×20	13.64	0.63	0.03	60	37
25×30	10.21	0.64	0.16	46	51
25×50	3.33	0.65	1.00	51	47
35×40	7.68	0.73	0.12	53	44
40×50	5.92	0.65	0.55	53	47
50×60	5.66	0.66	0.99	55	44

As we can see from the Tables, the modified bound is tight enough resulting in relative dual gap less than 1% for all classes of instances and all combination of $m \times n$. Concerning the quality of the primal bound, the relative gap is less than 1% for class C, less than 2% for class D, and around $5 - 7\%$ for class E. For the class C the greedy algorithm with f^1 slightly outperforms that with f^2, but for classes D and E, f^2,performs much better.

Table 2. Results for class D.

$m \times n$	gap_{lp}	gap_{dual}	gap_{prim}	f^1	f^5
8×10	18.09	0.96	0.08	20	61
18×20	17.91	0.64	0.31	32	59
25×30	13.05	0.56	1.31	33	56
25×50	4.87	1.00	1.00	4	13
35×40	13.02	0.69	1.04	34	62
40×50	10.10	0.73	1.66	21	64

50×60	9.89	0.72	1.69	25	69

To solve the modified dual problem we used the constraint generation technique (Benders scheme) which allows to calculate the value of the bound w_{MD} with prescribed precision. From practical point of view Benders scheme is slow, such that using another technique to solve the dual, e.g., bundle or volume methods, may be an interesting direction for future research.

Table 3. Results for class E.

$m \times n$	gap_{lp}	gap_{dual}	gap_{prim}	f^1	f^5
8×10	23.91	0.40	0.14	24	66
18×20	18.14	0.88	0.76	17	82
25×30	16.18	0.97	4.08	6	91
25×50	6.76	0.95	7.00	3	51
35×40	15.00	0.76	3.64	4	91
40×50	14.05	0.80	5.81	1	81
50×60	13.71	0.76	6.23	1	82

5. Conclusion

The modified Lagrangian bound was proposed for the generalized assignment problem. The approach was tested for problem instances where the number of jobs is relatively close to the number of agents. These instances are known to be difficult to solve due to the poor quality of the linear programming bound. Our computational study demonstrates that the modified dual bound is very tight for all types of benchmark GAP instances considered in the numerical experiment. Various heuristic techniques can be used to restore the feasibility of the Lagrangian solution. Usually a feasible solution is obtained only once in the course of the main algorithm, based on the optimal solution to the Lagrangian dual problem. Then a rather sophisticated and high-cost heuristic is implemented without raising significantly the overall cost of the solution technique. On the contrary, in our approach feasible solutions are obtained frequently, in all iterations of the technique used to solve the modified dual problem. The main idea is to move from a single use of a costly and hopefully more efficient technique to a multiple use of a low cost and maybe less efficient heuristic. Hence a low cost and simple technique is necessary to restore the feasibility. The proposed family of greedy heuristics meets these criteria and our computational experiment demonstrates that high quality feasible solutions can be generated this way.

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