TIME CONSISTENT RISK AVERSE DYNAMIC DECISION MODELS: AN ECONOMIC INTERPRETATION

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ABSTRACT

In this paper, we develop an economic interpretation for the objective function of time consistent risk-averse dynamic stochastic programming models with a recursive formulation. We illustrate the developed concepts using the portfolio selection problem with an widely-adopted risk measure, namely the Conditional Value-at-Risk, and show how to solve it for stage-wise independent returns.

KEYWORDS Time Consistency, Risk Averse Dynamic Stochastic Programming, Conditional Value-at-Risk.

Main area: PM - Mathematical Programming.

1 Introduction

Dynamic decision models under uncertainty are very common in financial planning and financial engineering problems. The major motivation on developing these models is the fact that we can incorporate the flexibility of dynamic decisions to improve our objective function. In other words, the possibility of changing a policy after the realization of some random variables improves the objective function and allows the first stage decisions to be less conservative than their counterpart in the static case. Hence, it doesn't make any sense to incorporate this flexibility if the future decisions are not actually going to be implemented.

In an optimization context, a multistage model is defined for each state, i.e., for each instant of time and uncertainty realization. Each model determines the current first-stage decision, which is the one actually implemented, and the future planned policy given by a sequence of decisions for each future state. As formally stated by Shapiro (2009) in definition (A1), this policy is time consistent iff the planned solutions coincides with the first-stage decisions of the future problems, for all states of the system. Thus, we can give the following interpretation:

> *A policy is time consistent if, and only if, the future planned decisions are actually going to be implemented.*

Moreover, if this sequence of multistage problems can be written recursively as stated by Shapiro (2009) in definition (A2), time consistency of the optimal policies is guaranteed by the Bellman's principle.

Note that Shapiro (2009) uses an indirect consequence of solving a sequence of recursive problems to develop the following interpretation of time consistency (see Section 1 of Shapiro (2009)) :

> "at every state of the system, our optimal decisions should not depend on scenarios which we already know cannot happen in the future"

Note also that time consistency can refer to a property of dynamic risk measures, as stated in Bion-Nadal (2008); Cheridito et al. (2006); Detlefsen and Scandolo (2005); Kovacevic and Pflug (2009); Riedel (2004); Roorda and Schumacher (2007); Ruszczynski (2010); Ruszczynski and Shapiro (2006). Indeed, this property is related to the definition of Shapiro (2009) since time consistent optimal policies can be obtained by choosing the objective functions of the multistage models to be time consistent dynamic risk measures. This modeling choice leads to a sequence of recursive problems where the Bellman's principle guarantees time consistent optimal policies (see definition A2 of Shapiro (2009) for details).

Furthermore, we argue that a time consistent policy is imperative since it is the only way to guarantee optimality of the implemented decisions. Indeed, the current implemented decision (first stage decision) is optimal if the future implemented ones are exactly the same as the optimal planned policy. Hence, if one chooses the recursive formulation for the objective functions then, time consistency and, consequently, optimality are guaranteed.

This recursive formulation is not commonly used in practice as a result of a lack of suitable economic interpretation for its objective function. Indeed, how can a decision maker choose a policy if he / she does not know what is actually going to be optimized?

For this reason, we prove that a particular set of these recursive objective functions can be interpreted as the certainty equivalent with respect to a time consistent dynamic utility defined as the recursive form of given (one-period) preference functionals. Moreover, recursive objective functions are the composed form of the certainty equivalents with respect to those functionals. In order to illustrate it, we propose a time consistent risk-averse dynamic stochastic programming model for the portfolio selection problem and interpret the results for this application.

2 Economic interpretation

Let us assume a multistage setting with a finite planning horizon T, where $\mathcal{H} = \{0, \ldots, T-1\}$. We consider the stochastic process $\mathbf{r_t}(\omega)$ and the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a related filtration $\mathcal{F}_0 \subseteq \ldots \subseteq \mathcal{F}_T$, where $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F} = \mathcal{F}_T$.

Let us consider a generic one-period preference functional $\psi_t : L^\infty(\mathcal{F}_{t+1}) \to L^\infty(\mathcal{F}_t)$ and, conditioned on a particular realization of the stochastic process $\bar{\mathbf{r}}_{[\mathbf{t}]} = [\mathbf{r_0}(\omega), \ldots, \mathbf{r_t}(\omega)]$, the related real valued function $\psi_t\left(\cdot \mid \bar{\mathbf{r}}_{[\mathbf{t}]}\right) : L^\infty(\mathcal{F}_{t+1}) \to \mathbb{R}$. Let us denote $(U_t)_{t\in\mathcal{H}}$ as the time consistent dynamic utility function generated by ψ_t (see Cheridito and Kupper (2009) for details). Formally speaking, $U_t : L^\infty(\mathcal{F}_T) \to L^\infty(\mathcal{F}_t)$ is defined as follows:

$$U_t(W_T) = \psi_t(U_{t+1}(W_T)), \quad \forall t \in \mathcal{H},$$

where $U_T(W_T) = W_T$ and $W_T \in L^\infty(\mathcal{F}_T)$.

Now, let us define the following dynamic stochastic programming model where the value function at time t depends on the decisions at $t-1$ and on the realization of the stochastic process until t.

Thus, for $t = T$ we define the value function

$$\mathcal{V}_T\left(\mathbf{x}_{T-1}, \bar{\mathbf{r}}_{[\mathbf{T}]}\right) = W_T\left(\mathbf{x}_{T-1}, \bar{\mathbf{r}}_{[\mathbf{T}]}\right),$$

where $W_T = W_T\left(\mathbf{x}_{T-1}, \bar{\mathbf{r}}_{[\mathbf{T}]}\right)$ is a real valued function.

For $t \in \mathcal{H} \setminus \{0\}$, we define the value function $\mathcal{V}_t\left(\mathbf{x}_{t-1}, \bar{\mathbf{r}}_{[\mathbf{t}]}\right)$ as

$$\sup_{\mathbf{x}_t \in \mathcal{X}_t} \psi_t\left(\mathcal{V}_{t+1}\left(\mathbf{x}_t, \mathbf{r}_{[\mathbf{t+1}]}\right) \,\middle|\, \bar{\mathbf{r}}_{[\mathbf{t}]}\right), \tag{1}$$

where $\mathcal{X}_t = \mathcal{X}_t\left(\mathbf{x}_{t-1}, \bar{\mathbf{r}}_{[\mathbf{t}]}\right)$.

For $t = 0$, we define the value function $\mathcal{V}_0$ as (1), where $\mathcal{X}_0 = X_0$. Then, we develop the following results.

Proposition 1. *If ψ_t is a translation invariant, monotone functional normalized to zero, then for $t \in \mathcal{H}$ the value function can be written as*

$$\mathcal{V}_t\left(\mathbf{x}_{t-1}, \bar{\mathbf{r}}_{[\mathbf{t}]}\right) = \sup_{\mathbf{x}_\tau \in \mathcal{X}_\tau, \forall \tau = t, \ldots, T-1} C_t\left(W_T \,\middle|\, , \bar{\mathbf{r}}_{[\mathbf{t}]}\right),$$

where $C_t\left(W_T \,\middle|\, \bar{\mathbf{r}}_{[\mathbf{t}]}\right)$ is the certainty equivalent of W_T w.r.t. U_t conditioned on the realization sequence $, \bar{\mathbf{r}}_{[\mathbf{t}]}$. See proof in A.

Corollary 2. *If ψ_t is a translation invariant, monotone functional normalized to zero, then for $t \in \mathcal{H}$ the value function $\mathcal{V}_t\left(\mathbf{x}_{t-1}, \bar{\mathbf{r}}_{[t]}\right)$ can be written as*

$$\sup_{x_\tau \in \mathcal{X}_\tau, \forall \tau = t, \ldots, T-1} \tilde{C}_t\left(\ldots \tilde{C}_{T-1}(W_T) \,\middle|\, \bar{\mathbf{r}}_{[t]}\right),$$

where $\tilde{C}_t$ and $\tilde{C}_t(\cdot \mid \bar{\mathbf{r}}_{[\mathbf{1,t}]})$ are the certainty equivalent w.r.t. ψ_t and $\psi_t(\cdot \mid \bar{\mathbf{r}}_{[\mathbf{t}]})$, respectively. See proof in B.

For sake of simplicity, we decided not to include intermediate "costs" as in Ruszczynski and Shapiro (2006), however we argue that our results would still hold true for that extension. It is worth mentioning that we define the feasible sets, $\mathcal{X}_t$, and the terminal function, $W_T\left(x_{T-1}, r_{[T]}\right)$ generically depending on the application. On the next section we define them to fit the portfolio selection problem.

3 A portfolio selection example

Let $\mathcal{A} = \{1, \ldots, A\}$ be the set of assets, $\mathcal{X}_t$ the set of feasible strategies and $W_T\left(\mathbf{x_{T-1}}, \bar{\mathbf{r}}_{[\mathbf{T}]}\right)$ the terminal wealth where $\mathbf{r_t}\left(\omega\right) = \left(r_{1,t}\left(\omega\right), \ldots, r_{A,t}\left(\omega\right)\right)$ and $\mathbf{x_t} = \left(x_{1,t}, \ldots, x_{A,t}\right)$ are asset returns and allocations, respectively.

Then, for $t > 0$ we define

$$\mathcal{X}_t\left(\mathbf{x}_{t-1}, \bar{\mathbf{r}}_{[\mathbf{t}]}\right) = \left\{\mathbf{x}_t \in \mathbb{R}^A : \sum_{i\in\mathcal{A}} x_{i,t} = \sum_{i\in\mathcal{A}} \left(1 + \bar{r}_{i,t}\right) x_{i,t-1}\right\}$$

and $\mathcal{X}_0 = \left\{x_0 \in \mathbb{R}^A : \sum_{i\in\mathcal{A}} x_{i,0} = W_0\right\}$, where W_0 is the initial wealth.

Moreover, we define the terminal wealth as

$$W_T(\mathbf{x}_{T-1}, \bar{\mathbf{r}}_{[\mathbf{T}]}) = \sum_{i\in\mathcal{A}} \left(1 + \bar{r}_{i,T}\right) x_{i,T-1}$$

and choose the (one-period) translation invariant, monotone and normalized utility functional ψ_t to be the convex combination of the expected value and the CVaR based acceptability measure, formally defined as

$$\psi_t\left(\mathcal{V}_{t+1}\right) = \left(1-\lambda\right)\mathbb{E}\left[\mathcal{V}_{t+1} \mid \mathbf{r}_{[\mathbf{t}]}\right] + \lambda\,\phi_t^\alpha\left(\mathcal{V}_{t+1}, \mathbf{r}_{[\mathbf{t}]}\right), \tag{2}$$

where $\lambda \in [0,1]$, $\mathcal{V}_{t+1} = \mathcal{V}_{t+1}\left(\mathbf{x_t}, \mathbf{r}_{[\mathbf{t+1}]}\right)$ and

$$\phi_t^\alpha\left(\mathcal{V}_{t+1}, \mathbf{r}_{[t]}\right) = \sup_{z\in\mathbb{R}}\left\{z - \frac{\mathbb{E}\left[\left(\mathcal{V}_{t+1} - z\right)^- \mid \mathbf{r}_{[\mathbf{t}]}\right]}{1-\alpha}\right\}.$$

Following Rockafellar and Uryasev (2000), we rewrite (1) as the following dynamic stochastic programming problem:

$$\begin{array}{ll} \underset{\mathbf{x}_t}{\text{maximize}} & \mathbb{E}\left[\left(1-\lambda\right)\mathcal{V}_{t+1} + \lambda\left(z - \dfrac{\left(\mathcal{V}_{t+1} - z\right)^-}{1-\alpha}\right) \middle| \, \bar{\mathbf{r}}_{[\mathbf{t}]}\right] \\ \text{subject to} & \sum_{i\in\mathcal{A}} x_{i,t} = \sum_{i\in\mathcal{A}} \left(1 + r_{i,t}\right) x_{i,t-1} \\ & \mathbf{x_t} \geq 0, \end{array}$$

where $\mathcal{V}_{t+1}$ stands for $\mathcal{V}_{t+1}\left(\mathbf{x_t}, \mathbf{r}_{[\mathbf{t+1}]}\right)$.

The objective function of the proposed model at t is the certainty equivalent with respect to the time consistent dynamic utility function generated by the one-period preference functionals of the investor. This recursive formulation ensures time consistent optimal policies and it is motivated by Corollary 2. The objective at $t = T-1$ is to maximize the certainty equivalent (CE) of terminal wealth with respect to the one-period preference functional ψ_{T-1}. Indeed, we can interpret the optimal CE as the portfolio value since it is the deterministic amount of money the investor would accept instead of the (random) terminal wealth obtained by his / her optimal trading strategy. At $t = T-2, \ldots, 0$, the preference functional ψ_t is applied to the (random) portfolio value whose realizations are given by all possible optimal CE's at $t+1$. Thus, the objective at time t is to maximize the CE of the future (random) portfolio value and, consequently, the value function at t is the current portfolio value for that particular investor.

In order to have an illustrative example, let us assume a probability space to be represented by a discrete event tree, where the scenarios $\omega \in \Omega$ are numbered by the terminal nodes. Let us denote an intermediate node N as a subset of Ω and $\mathcal{N}_t$ as the set of all nodes at stage t, i.e., the unique partition that generates the σ-algebra $\mathcal{F}_t$, $\forall t \in \mathcal{H} \cup \{T\}$. For sake of simplicity, consider the event tree represented in Figure 1, where $\Omega = \{1, 2, 3, 4\}$. In our notation, the root node is defined as $\Omega = \{1, 2, 3, 4\}$, the intermediate nodes as $\{1, 2\}$ and $\{3, 4\}$ and the terminal nodes as $\{1\}, \{2\}, \{3\}, \{4\}$. In addition, we have that $\mathcal{N}_1 = \{\Omega\}$, $\mathcal{N}_2 = \{\{1, 2\}, \{3, 4\}\}$ and $\mathcal{N}_3 = \{\{1\}, \{2\}, \{3\}, \{4\}\}$.

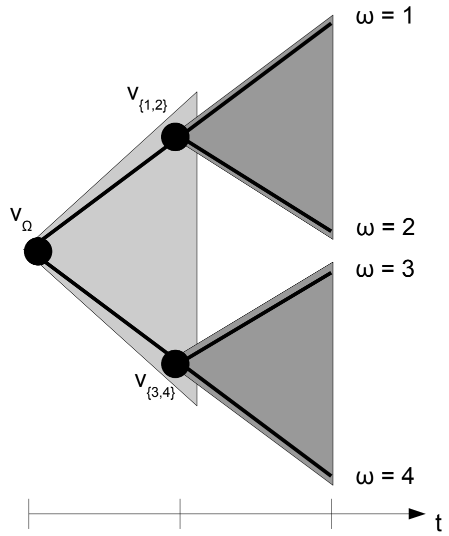

Figure 1: Event tree

Considering Figure 1, the current portfolio value is given by $\mathrm{v}_\Omega = \mathcal{V}_0$ and the (random) portfolio value at $t = 1$ is given by the realizations

$$\mathrm{v}_{\{1,2\}} = \mathcal{V}_1\left(\mathbf{x_0}, \bar{\mathbf{r}}_{[\mathbf{1}]}^{\{1,2\}}\right) \text{ and } \mathrm{v}_{\{3,4\}} = \mathcal{V}_1\left(\mathbf{x_0}, \bar{\mathbf{r}}_{[\mathbf{1}]}^{\{3,4\}}\right),$$

where $\bar{\mathbf{r}}_{[\mathbf{1}]}^{N} = \mathbf{r}_{[\mathbf{1}]}(\omega)$, such that $\omega \in N$.

From Corollary 2, the current portfolio value v_Ω is the optimal certainty equivalent of the (random) portfolio value at $t = 1$. Moreover, from Proposition 1, v_Ω is the optimal certainty equivalent for an investor with the time consistent dynamic utility function generated by (2).

In particular, one can easily solve this problem for stagewise independent returns. To do so, let us define the intermediate wealth $W_{t+1} = \sum_{i\in\mathcal{A}} (1 + r_{i,t+1})\, x_{i,t}$, $\forall t \in \mathcal{H}$, and an equivalent value function $V_t(W_t)$ as

$$\begin{aligned} \underset{W_{t+1}, \mathbf{x}_t, z}{\text{maximize}} \quad & \mathbb{E}\left[(1-\lambda)\, V_{t+1} + \lambda\left(z - \frac{(V_{t+1} - z)^{-}}{1-\alpha}\right)\right] \\ \text{subject to} \quad & W_{t+1} = \textstyle\sum_{i\in\mathcal{A}} (1 + r_{i,t+1})\, x_{i,t} \\ & \textstyle\sum_{i\in\mathcal{A}} x_{i,t} = W_t \\ & \mathbf{x}_t \geq 0, \end{aligned} \tag{3}$$

where V_{t+1} stands for $V_{t+1}(W_{t+1})$ and all constraints represented are defined for almost every $\omega \in \Omega$, i.e., in the $\mathbb{P}$ a.s. sense.

Following Blomvall and Shapiro (2006), problem (3) has a myopic optimal policy which can be obtained as the solution sequence ($\forall t \in \mathcal{H}$) of the following two-stage problem:

$$\begin{array}{ll} \underset{W_{t+1},\mathbf{x}_t,z}{\text{maximize}} & \mathbb{E}\left[(1-\lambda)\, W_{t+1} + \lambda \left(z - \dfrac{(W_{t+1}-z)^-}{1-\alpha}\right)\right] \\ \text{subject to} & W_{t+1} = \sum_{i\in\mathcal{A}} (1+r_{i,t+1})\, x_{i,t} \\ & \sum_{i\in\mathcal{A}} x_{i,t} = W_t \\ & \mathbf{x}_t \geq 0. \end{array} \tag{4}$$

For an discrete distribution, we obtain the optimal policy, i.e, the optimal solution for each time step $t \in \mathcal{H}$ and for each node $N \in \mathcal{N}_t$, by solving the deterministic equivalent linear program

$$\begin{array}{ll} \underset{q,W_{t+1},\mathbf{x}_t,z}{\text{maximize}} & \sum\limits_{\omega\in N} \mathbb{P}(\omega)\left[(1-\lambda)\, W_{t+1}(\omega) + \lambda\left(z - \dfrac{q(\omega)}{1-\alpha}\right)\right] \\ \text{subject to} & W_{t+1}(\omega) = \sum_{i\in\mathcal{A}} (1+r_{i,t+1}(\omega))\, x_{i,t}, \quad \forall\, \omega \in \Omega \\ & \sum_{i\in\mathcal{A}} x_{i,t} = W_t(N) \\ & q(\omega) \geq z - W_{t+1}(\omega), \quad \forall \omega \in \Omega \\ & q(\omega) \geq 0, \quad \forall \omega \in \Omega \\ & \mathbf{x}_t \geq 0. \end{array}$$

Hence, the proposed model is easily solved and strongly motivated by the economic interpretation developed in Proposition 1 and Corollary 2.

4 Conclusions

In this paper, we develop a suitable economic interpretation for a particular set of time consistent risk-averse dynamic problems based on a recursive objective function. We prove that the objective function is the certainty equivalent with respect to the time consistent dynamic utility function defined as the composed form of one-period preference functionals. We prove that this objective is the composed form of certainty equivalents with respect to these one-period preference functionals. These results motivate time consistent models as long as they are the only choice that guarantees optimality of the implemented decisions.

In addition, we develop a time consistent dynamic stochastic programming model for portfolio selection in which the objective function is a recursive setting of a convex combination between expectation and (negative of) CVaR applied to terminal wealth. The developed economic interpretation gives us the intuition that at stage t the agent is maximizing the certainty equivalent of the portfolio value with respect to his / her one-period preference functional and that the value function at t is the portfolio value for that investor. We use an example to illustrate the developed concepts and to motivate future applications of this recursive formulation.

A Proof of Proposition 1

Proof. By definition we have

$$\begin{aligned}\mathcal{V}_t\left(\mathbf{x}_{t-1}, \bar{\mathbf{r}}_{[\mathbf{t}]}\right) &= \sup_{\mathbf{x}_t \in \mathcal{X}_t} \psi_t\left(\mathcal{V}_{t+1}\left(\mathbf{x}_t, \bar{\mathbf{r}}_{[\mathbf{t+1}]}\right) \mid \bar{\mathbf{r}}_{[\mathbf{t}]}\right) \\ &= \sup_{\mathbf{x}_t \in \mathcal{X}_t} \psi_t\left(\ldots \sup_{\mathbf{x}_{T-1} \in \mathcal{X}_{T-1}} \psi_{T-1}\left(W_T\right) \middle| \bar{\mathbf{r}}_{[\mathbf{t}]}\right).\end{aligned}$$

Using the monotonicity of ψ_t and the definition of U_t we have the following:

$$\begin{aligned}&\mathcal{V}_t\left(\mathbf{x}_{t-1}, \bar{\mathbf{r}}_{[\mathbf{t}]}\right) \\ &= \sup_{\mathbf{x}_\tau \in \mathcal{X}_\tau, \forall \tau = t, \ldots, T-1} \psi_t\left(\ldots \psi_{T-1}\left(W_T\right) \mid \bar{\mathbf{r}}_{[\mathbf{t}]}\right) \\ &= \sup_{\mathbf{x}_\tau \in \mathcal{X}_\tau, \forall \tau = t, \ldots, T-1} U_t\left(W_T \mid \bar{\mathbf{r}}_{[\mathbf{t}]}\right).\end{aligned} \tag{5}$$

By the certainty equivalent definition we have that $C_t\left(W_T \mid \bar{\mathbf{r}}_{[\mathbf{t}]}\right)$ satisfies $U_t\left(C_t\left(W_T \mid \bar{\mathbf{r}}_{[\mathbf{t}]}\right) \mid \bar{\mathbf{r}}_{[\mathbf{t}]}\right) = U_t\left(W_T \mid \bar{\mathbf{r}}_{[\mathbf{t}]}\right)$. It is easy to show that $U_t\left(\cdot \mid \bar{\mathbf{r}}_{[\mathbf{t}]}\right)$ is translation invariant and normalized to zero, since its generators ψ_t have the same properties. Then, $U_t\left(C_t\left(W_T \mid \bar{\mathbf{r}}_{[\mathbf{t}]}\right) \mid \bar{\mathbf{r}}_{[\mathbf{t}]}\right) = C_t\left(W_T \mid \bar{\mathbf{r}}_{[\mathbf{t}]}\right)$ and consequently, $U_t\left(W_T \mid \bar{\mathbf{r}}_{[\mathbf{t}]}\right) = C_t\left(W_T \mid \bar{\mathbf{r}}_{[\mathbf{t}]}\right)$.

Finally we have that

$$\mathcal{V}_t\left(\mathbf{x}_{t-1}, \bar{\mathbf{r}}_{[\mathbf{t}]}\right) = \sup_{\mathbf{x}_\tau \in \mathcal{X}_\tau, \forall \tau = t, \ldots, T-1} C_t\left(W_T \mid \bar{\mathbf{r}}_{[\mathbf{t}]}\right).$$

□

B Proof of Corollary 2

Proof. By the certainty equivalent definition we have that $\tilde{C}_t\left(\cdot \mid \bar{\mathbf{r}}_{[\mathbf{t}]}\right)$ satisfies $\psi_t\left(\tilde{C}_t\left(\cdot \mid \bar{\mathbf{r}}_{[\mathbf{t}]}\right) \mid \bar{\mathbf{r}}_{[\mathbf{t}]}\right) = \psi_t\left(\cdot \mid \bar{\mathbf{r}}_{[1,t]}\right)$ and using the assumption that ψ_t is translation invariant and normalized to zero, we have $\psi_t = \tilde{C}_t$. Note that this property holds true for the conditional version. Then, from equation (6) we have the following:

$$\begin{aligned}&\mathcal{V}_t\left(\mathbf{x}_{t-1}, \bar{\mathbf{r}}_{[\mathbf{t}]}\right) \\ &= \sup_{\mathbf{x}_\tau \in \mathcal{X}_\tau, \forall \tau = t, \ldots, T-1} \psi_t\left(\ldots \psi_{T-1}\left(W_T\right) \mid \bar{\mathbf{r}}_{[\mathbf{t}]}\right) \\ &= \sup_{\mathbf{x}_\tau \in \mathcal{X}_\tau, \forall \tau = t, \ldots, T-1} \tilde{C}_t\left(\ldots \tilde{C}_{T-1}\left(W_T\right) \middle| \bar{\mathbf{r}}_{[\mathbf{t}]}\right).\end{aligned}$$

□

References

Bion-Nadal, J., 2008. Dynamic risk measures: Time consistency and risk measures from BMO martingales. Finance and Stochastics 12 (2), 219–244.

Blomvall, J., Shapiro, A., 2006. Solving multistage asset investment problems by the sample average approximation method. Mathematical Programming 108 (2), 571–595.

Cheridito, P., Delbaen, F., Kupper, M., 2006. Dynamic monetary risk measures for bounded discrete-time processes. Electronic Journal of Probability 11 (3), 57–106.

Cheridito, P., Kupper, M., 2009. Recursiveness of indifference prices and translation-invariant preferences. Mathematics and Financial Economics 2 (3), 173–188.

Detlefsen, K., Scandolo, G., 2005. Conditional and dynamic convex risk measures. Finance and Stochastics 9 (4), 539–561.

Kovacevic, R., Pflug, G., 2009. Time consistency and information monotonicity of multiperiod acceptability functionals. Radon Series Comp. Appl. Math 8, 1–24.

Riedel, F., 2004. Dynamic coherent risk measures. Stochastic Processes and their Applications 112 (2), 185–200.

Rockafellar, R. T., Uryasev, S., 2000. Optimization of conditional value-at-risk. Journal of risk 2, 21–42.

Roorda, B., Schumacher, J., 2007. Time consistency conditions for acceptability measures, with an application to tail value at risk. Insurance: Mathematics and Economics 40 (2), 209–230.

Ruszczynski, A., 2010. Risk-averse dynamic programming for markov decision processes. Mathematical Programming 125 (2), 235–261.

Ruszczynski, A., Shapiro, A., 2006. Conditional risk mappings. Mathematics of Operations Research 31 (3), 544–561.

Shapiro, A., 2009. On a time consistency concept in risk averse multistage stochastic programming. Operations Research Letters 37 (3), 143–147.