A Model to the Ellipsoidal Covering Problem

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Resumo

O problema de recobrimento elipsoidal consiste em cobrir um elipsóide com esferas cujos raios pertencem a um conjunto discreto. A natureza discreta dos raios das esferas é uma das dificuldades inerentes a este problema quando tentamos resolvê-lo e outra dificuldade é garantir que cada ponto do elipsóide seja coberto por pelo menos uma esfera. Apesar destas dificuldades, uma boa razão para o estudo deste problema é a sua aplicação na configuração de máquinas de raios gama, que são utilizadas em tratamentos de tumores cerebrais. Este é um problema semi-infinito linear discreto e apresentamos uma versão fraca do mesmo que usa a idéia do Problema de Localização de Facilidades para determinar uma posição mais provável para os centros das esferas de forma que cubram todo o elipsóide.

Palavras Chave: Otimização Discreta,Otimização Global, Gamma Knife

Abstract

The ellipsoidal covering problem consists in covering an ellipsoid with spheres whose radii belongs to a discrete set. The discrete nature of the radii of the spheres is one of the difficulties inherent to this problem when one tries to solve it and another dificulty is ensuring that every point of the ellipsoid is covered by at least one sphere. Despite these difficulties, a good reason to study this problem is its application in configuring gamma ray machines, used in brain tumors treatments. This problem is a semi-infinite nonlinear discrete one and we present a weak version that uses the idea of the Facility Location Problem to determine a likely location for the center of the spheres in such a way that the ellipsoid must be covered by them.

Keywords: Discrete Optimization, Global Optimization, Gamma Knife

1 Introduction

The discrete problem of ellipsoidal covering has an application in configuring Gamma ray machines. These machines are used in stereotactic radiation therapy to brain tumors treatments. It delivers "shots" that are extremely precise, reaching the tumor area in a shape of spheres. Due to the use of multiple shots centered at the disease area, the healthy tissue receives minimal dose of radiation. In the other hand, the use of multiple shots can result in exposing the tumor to a higher dose of radiation which happens when the spheres are superimposed, and such action must not happen. All the tumor area should be covered homogeneously by the treatment. In order to achieve this goal we must define the number of shots that have to be done as well as its positions and dosages (see [1] and [3]). This task nowadays demands time and a lot of experience and knowledge from the person that is planning the treatment and it makes the treatment very expensive. By automating the treatments planning these costs may decrease and more people can access it.

We present in this work a new model for the discrete problem of ellipsoidal covering that uses the idea of the Facility Location Problem [4] to determine the probable positions for the centers of the spheres in such a way that each sphere is like a facility that attends parts of the ellipsoid. Given an ellipsoid with (x_0, y_0, z_0) as center coordinates and R_x, R_y, R_z as its radii and a set of radii of spheres, $r \in \{r_1, r_2, r_3 ..., r_M\}, r < \min\{R_x, R_y, R_z\}$, the problem is to cover the ellipsoid with spheres. There are two peculiarities that makes this problem a discrete one: the radii of the spheres belongs to the set above and the number of spheres that must be integer. For this reason, most of the existent approaches to solve this problem are based in discrete optimization techniques. We want to emphasize that our goal is to solve the ellipsoidal covering problem, which differs from the complete solution for the configuration of Gamma ray machines in the sense that it is not in our scope the determination of the dosage of each shot (see [1] and [3]).

In this work, the model proposed is totally integer, but differs from the integer models presented in [1]. One of the differences is that in [1], a fixed cubic lattice of m^3 points in $\mathcal{R}^3$ is used to discretize the entire ellipsoid and here a mesh is placed on the border of the ellipsoid and the points that are inside of the ellipsoid are points of Weber, which are calculated using the coordinates of the points that are on the mesh. This work also differs from [2] because the model is totally different and we do not use Geometric Programming to solve it.

The text is organized like that: in section 2 we present in detail the Discret Ellipsoidal Covering Problem (DECP) and the Weak Discret Ellipsoidal Covering Problem (WDECP) and a theoretical basis to solve the first (DECP) using the second one (WDECP). Besides that, we present a model for the Discret Ellipsoidal Covering Problem (DECP). In section 3 the computational results are presented.

Terminology:
$c = (x_0, y_0, z_0)$ is the center of the ellipsoid;
$C(w, r)$ is the cube of center w and inscribed in the sphere of radius r;
$D(w, r)$ is the dodecahedron of center w and inscribed in the sphere of radii r;
$E(c, R)$ is the ellipsoid of center c and radii (R_x, R_y, R_z), where $R = diag(R_x, R_y, R_z)$;
γ is the level of intersection between two different spheres;
r_i is the radius of the i-sphere, $i = 1, \ldots, n$;
R is the matrix which diagonal are the radii of the ellipsoid;
$\{R_x, R_y, R_z\}$ are the radii of the ellipsoid;
$S(w, r)$ is the sphere of center w and radius r;
θ is the ratio between the volume of a sphere and the volume of the inscribed cube or dodecahedron;

$\|v\|$ is the euclidean norm; $\sqrt{\sum_{i=1}^{n} v_i^2}$;
$\|v\|_\infty$ is the maximum norm; $\|v\|_\infty = max\{|v_1|, \ldots, |v_n|\}$;
$Vol(S)$ is the volume of solid S;
$w_i = (w_i^x, w_i^y, w_i^z)$ is the center of the ith-sphere;
$\aleph_0(C)$ is the cardinality of the set C;
$\overline{R} = \max\{R_x, R_y, R_z\}$.

2 The discrete ellipsoidal covering problem - DECP

The goal of this section is to detail the DECP. This problem consists in covering an ellipsoid with spheres. It is more specifically defined below:

Given $(R_x, R_y, R_z) \in \mathcal{R}^3_{++}, c \in \mathcal{R}^3, n \in N$, the ellipsoid of center c and radii (R_x, R_y, R_z) is defined by the following set:

$$E(c, R) = \left\{w \in \mathcal{R}^3; (w-c)^t R^{-2}(w-c) \leq 1\right\}, \tag{1}$$

where $R = diag(R_x, R_y, R_z)$.

We can define DECP as:

Definition 1. *Given an ellipsoid E(c,R) a* ***discrete ellipsoidal covering*** *(DEC) is a structure of the form:*

$$Pell(E) = \{\boldsymbol{C}, \boldsymbol{r}\}, \tag{2}$$

where $\boldsymbol{C} = \{w_1, w_2, \ldots, w_n\}$, $\boldsymbol{r} = \{r_i \in \{r_1, r_2, \ldots, r_M\},\ i = 1, \ldots, n\}$, w_i and r_i satisfy the following conditions:

1. *$w_i \in E(c, R)$ for all n;*
2. *if $w \in E(c, R)$ then $\|w - w_i\| \leq r_i$ for some $i = 1, \ldots, n$;*
3. *the number of spheres n must be as small as possible.*

$\boldsymbol{C}$ and $\boldsymbol{r}$ are respectively the set of the centers of the spheres and the set of the discrete radii. If $\boldsymbol{r} = \{r \in [a, b]\}$ the covering is said to be continuous. The discrete ellipsoidal covering problem (DECP) can be seen as determining a pair $P = \{\boldsymbol{C},\ \boldsymbol{r}\}$.

The Definition 1 suggests that we treat the $(DECP)$ as a viability problem with semi infinite restrictions. To avoid this semi infinite characteristic, we propose a formulation named Weak Discrete Ellipsoidal Covering $(WDEC)$, presented in Definition 2.

Definition 2. *Given an ellipsoid E(c,R) a* ***weak discrete ellipsoidal covering****(WDEC) is a structure of the form:*

$$\underline{Pell}(E(c, R)) = \{\boldsymbol{C}, \boldsymbol{r}\},$$

where $\boldsymbol{C} = \{w_1, w_2, \ldots, w_n\}$ e $\boldsymbol{r} = \{r_i \in \{r_1, r_2, \ldots, r_M\},\ i = 1, \ldots, n\}$, w_i, r_i and r_j satisfy the following conditions:

1. *$w_i \in E(c, R)$ for all n;*
2. *$\|w_i - w_j\| \geq \gamma(r_i + r_j)$ for all $i = 1, \ldots, n,\ j = i+1, \ldots, n,$ $r_i, r_j \in \{r_1, r_2, \ldots, r_M\}$;*
3. *The number of spheres n must be as small as possible.*

$\boldsymbol{C}$ and $\boldsymbol{r}$ are respectively the set of the centers of the spheres and the set of the discrete radii, the parameter γ is such that γr is the radius of the sphere that is inscribed in the cube ($\gamma = \frac{1}{\sqrt{3}}$) or in the dodecahedron ($\gamma = \frac{\sqrt{10(25+11\sqrt{5})}}{5\sqrt{3}(1+\sqrt{5})}$) that is inscribed in $S(w,r)$. Mathematically we have: $S(w,\gamma r) \subset C(w,r) \subset S(w,r)$

Both of the above structures differ only with respect to item 2. In Definitions 1 and 2 at ($WDEC$) we relax the condition of total covering given by item 2 of Definition 1 and obtain a weaker condition that is given by item 2 of Definition 2 and aiming to solve the viability problem in Definition 1 we propose to solve a maximization problem whose constraints satisfies items 1,2 and 3 of Definition 2 besides some aditional constraint that ensures the condition of total covering given by item 2 of Definition 1.

Hereafter, our work will be about finding a weak ellipsoidal covering for a given ellipsoid E_{seg} such that $E(c,R) \subset E_{seg}$ with $E_{seg} = E(c, R_{seg})$, whose radii of the spheres are as big as possible and impose the following condition:

Condition 1. *If $S(w_i, r_i)$ is the i-sphere with center w_i and radius r_i then $S(w_i, r_i) \subset E_{seg}$.*

E_{seg}, will be called **security ellipsoid** and is defined as:

Definition 3. *Given an ellipsoid $E(c,R)$ and $\epsilon > 0$ we define the security ellipsoid E_{seg} as:*

$$E_{seg} = \left\{ w \in \mathcal{R}^3; (w-c)^t R_{seg}^{-2} (w-c) \leq 1 \right\},$$

where $R_{seg} = (1+\epsilon) diag\,(R_x, R_y, R_z)$.

The Condition 1 is not essential because it can be obtained in an indirect way as Proposition 1 shows.

Proposition 1. *Given $\epsilon > 0, r > 0, c \in \mathcal{R}^3$ and $(R_x, R_y, R_z) \in \mathcal{R}^3_{++}$. Let $R_{min} = min\{R_x, R_y, R_z\}$, $r_{max} = max\{\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_M\}$, $\widetilde{\epsilon} = (1 - \epsilon\frac{R_{min}}{r_{max}})$ and $R_{\widetilde{\epsilon}} = diag(R_x - \widetilde{\epsilon} r, R_y - \widetilde{\epsilon} r, R_z - \widetilde{\epsilon} r)$. If $r \leq r_{max} \leq R_{min}$ and $w \in E(c, R_{\widetilde{\epsilon}})$ then $S(w,r) \subset E_{seg}$.*

Proof 1. *Given $p \in S(w,r)$,*

$$(p-c)^t R_{seg}^{-2} (p-c) \leq \frac{\overline{R} + (1-\widetilde{\epsilon})r}{\overline{R}(1+\epsilon)} = \frac{\overline{R} + \epsilon r \frac{R_{min}}{r_{max}}}{\overline{R}(1+\epsilon)} \leq 1.$$

Let $B_\infty[0,1] = \{v \in \mathcal{R}^3 |\, \|v\|_\infty = 1\}$ and $Z = \{x_1, x_2, \ldots, x_n\} \subset B_\infty[0,1]$. If

$$\|x_i - x_j\| \leq \frac{2(r-1)}{\overline{R}}$$

for some i, j, where r is the radius of the largest sphere contained in $E(c,R)$ then we have the following results.

Proposition 2. *If $x \in B_\infty[0,1]$, then*

$$w \in \partial E(0,R) = \{x \in \mathcal{R}^3 | x^t R^{-2} x = 1, \quad R = diag(R_x, R_y, R_z)\}.$$

Proof 2. $\frac{xR}{\|x\|} R^{-2} \frac{Rx}{\|x\|} = 1.$

Proposition 3. *If $\|x_i - x_j\| \geq \frac{2(r-1)}{\overline{R}}$ then $\|w_i - w_j\| \leq 2(r-1)$.*

Proof 3. $\|w_i - w_j\| = \left\| \frac{Rx_i}{\|x_i\|} - \frac{Rx_j}{\|x_j\|} \right\| \leq \overline{R} \left\| \frac{x_i}{\|x_i\|} - \frac{x_j}{\|x_j\|} \right\|.$
But

$$\left\| \frac{x_i}{\|x_i\|} - \frac{x_j}{\|x_j\|} \right\| \leq \frac{\overline{r}}{r} \|x_i - x_j\| \leq \|x_i - x_j\|,$$

where

$$\overline{r} = \|\overline{w}\| - r \text{ with } \overline{w}_i = x_i \text{ and } \overline{w}_j = \frac{x_j}{\|x_i\|} \quad \text{if} \quad \|x_i\| \leq \|x_j\|$$

or

$$\overline{w}_i = \frac{x_i}{\|x_j\|} \text{ and } \overline{w_j} = x_j \quad \text{if} \quad \|x_j\| \leq \|x_i\|.$$

As we have seen before, $\|x_i - x_j\| \leq \frac{2(r-1)}{\overline{R}}$ *and therefore*

$$\|w_i - w_j\| \leq \overline{R}\frac{2(r-1)}{\overline{R}} = 2(r-1).$$

This proposition means that given two points inside the ellipsoid, there exists at least one that satisfies $\|w_i - w_j\| \leq 2(r-1)$.

Definition 4. *For each* $i = 1, \ldots, N$ *where* $N = \aleph_0(Z)$, $Z = \{x_1, x_2, \ldots, x_n\} \subset B_\infty[0,1]$, *let*

$$W_i = \{w_j; \|w_i - w_j\| \leq 2(r-1)\},\ N_i = \aleph_0(z_i),$$

$$\overline{w}_i = \frac{1}{N_i}\left(\sum_{j=1}^{N_i} w_j^1,\ \sum_{j=1}^{N_i} w_j^2,\ \sum_{j=1}^{N_i} w_j^3\right)^t.$$

The points $\overline{w}_i$ *are called barycentre or Weber points of* W_i. *The center of the ellipsoid also belongs to this set so it has* $N+1$ *points.*

Definition 5. *Let* $\overline{w}_i$ *be the facility* i, w_j *be the location* j,
$\delta_{ij} = 1$, *if* $\overline{w_i}$ *attends* w_j *or* $\delta_{ij} = 0$, *otherwise;*
$z_{ij} = 1$, *if* $S(\overline{w}_i, r_i)$ *and* $S(\overline{w}_j, r_j)$ *intercept each other or* $z_{ij} = 0$, *otherwise.*
$y_i = 1$, *if* $\overline{w}_i$ *is the center of the ith sphere or* $y_i = 0$ *otherwise;*
$\lambda_i = 1$, *if the sphere* $S(\overline{w}_i, r_i)$ *has center* $\overline{w}_i$ *and radi* r_i *or* $\lambda_i = 0$, *otherwise.*
The optimization problem associated to <u>Pell</u>*(E(c,R)) that uses the idea of the Facility Location problem is given by:*

$$\begin{array}{lll}
\text{Minimize} & \displaystyle\sum_{i=1}^{N+1}\sum_{j=1}^{N} D_{i(j+N+1)}\delta_{ij} - \sum_{i=1}^{N+1}\sum_{j=1}^{N+1} D_{ij}z_{ij} + \sum_{i=1}^{N+1} r_i + M\sum_{i=1}^{N+1} y_i & \\
\text{Subject to :} & \displaystyle\sum_{j=1}^{N}\delta_{ij} + \sum_{j=1}^{N+1} z_{ij} - (N+1)y_i \leq 0, & (3)\\
& \displaystyle\sum_{i=1}^{N+1}\delta_{ij} \geq 1, & (4)\\
\text{P1} & \displaystyle\sum_{i=1}^{N+1} z_{ij} \geq 1, & (5)\\
& 2\delta_{ij}r_{max} - r_i \leq 2r_{max} - D_{i(j+N+1)} & (6)\\
& 2\overline{R}z_{ij} - r_i - r_j \leq 2\overline{R} - D_{ij}, & (7)\\
& 4r_{max}z_{ij} + 2(r_i + r_j) \leq 3D_{ij} + 4r_{max}, & (8)\\
& r_i \leq r_{max}y_i & (9)\\
& r_i - 2\lambda_2 + 4\lambda_3 + 7\lambda_4 + 9\lambda_5 = 0, & (10)\\
& \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 = 1, & (11)\\
& \delta_{ij} \in \{0,1\}, z_{ij} \in \{0,1\}, y_i \in \{0,1\}, \lambda_i \in \{0,1\}. & (12)
\end{array}$$

In the objective function, D_{ij} is the distance between the barycentre and the points at the border and M is a big constant. In the constraints, R_x is the larger radi of the ellipsoid and r_{max} is the bigger possible value for a radii of a sphere in a given ellipsoid. This model is divided in three groups of constraints: the first group comprises the constraints (3) and (4) that are related to covering, the ones from (5) to (9) are on the second group and are related to the intersections between the spheres and the last group is formed by constraints (10) and (11) that sets the radi of each sphere. The constraint (3) means that it is possible that each barycentre cover all border points and have intersections with other barycenters. The constraint (4) assures that each border point will be covered by at least one barycentre. Constraint (5) estabilishes that one barycenter has to intercept at least one other. Constraint (7) makes the spheres cover the point(s) of the border that are closer. Constraints (8) and (9) estabilishes that two spheres can have just one point of intersection or have some intersection without being superimposed or they do not have any intersection. Constraint (10) together with (11) and (12) estabilishes one radii among $\{0, 2, 4, 7, 9\}$ for each sphere. If the sphere is not selected its radi is 0. It is noteworthy that this model is convex.

2.1 Evaluation of the level of covering in a weak discrete ellipsoidal covering (WDEC)

Here we try to evaluate if the (WDEC) can provide a good covering and give a simple condition to have $\underline{Pell}(E(c,R)) = Pell(E(c,R))$, that is, the weak discrete ellipsoidal covering equals to the discrete ellipsoidal covering. Aiming at this purpose we build the results and definitions that are in this section.

Definition 6. *Given $d > 0$, an ellipsoid $E(c,R)$ and $\underline{Pell}(E(c,R)) = (\boldsymbol{C},\boldsymbol{r})$ and a (WDEC) for $E(c,R)$ we define:*

- *A mesh for $E(c,R)$ is the intersection: $E(c,R) \cap M(d)$ where:*

  $$M(d) = \{w \in \mathcal{R}^3; w = c + (-R_x : d : R_x, -R_y : d : R_y, -R_z : d : R_z)\}.$$

  This means that the mesh is generated accordingly to each (R_x, R_y, R_z) of the ellipsoid and the distance d between its points will determine if there will be more or less points at the mesh.

- *The level of covering of a (WDEC) $\underline{Pell}(E(c,R)) = (\boldsymbol{C},\boldsymbol{r})$ is defined by:*

  $$IP(\underline{Pell}(E(c,R))) = \frac{\aleph_0(\underline{Pell}(E(c,R) \cap M(d)))}{\aleph_0(M(d))},$$

  where: $\underline{Pell}(E(c,R) \cap M(d)) = \{w \in M(d) \cap S(\tilde{w},\tilde{r})\}$ for some pair $(\tilde{w},\tilde{r}) \in (\boldsymbol{C},\boldsymbol{r})$.

- *We say that a (WDEC) $\underline{Pell}(E(c,R)) = (\boldsymbol{C},\boldsymbol{r})$ of $(E(c,R))$ is total or perfect if $IP(\underline{Pell}(E(c,R))) = 1$.*

Proposition 5 gives us a characterization between the ellipsoidal covering and the weak ellipsoidal covering.

Proposition 4. *Given $d > 0$, a mesh $M(d)$, a (WDEC) $\underline{Pell}(E(c,R)) = (\boldsymbol{C},\boldsymbol{r})$ for $E(c,R)$ and set $\underline{Pell_d}(E(c,R)) = (C, r_d)$ where $r_d(i) = r(i) - d\sqrt{3}, i = 1, \ldots, n$, then*

$$\underline{Pell_d}(E(c,R)) = Pell_d(E(c,R)) \text{ if and only if } IP(\underline{Pell_d}(E(c,R))) = 1.$$

Proof 4. *If* $\underline{Pell}_d(E(c,R)) = Pell_d(E(c,R))$ *then for all* $v \in E(c,R), \exists S(w,r)$ *so that* $\|v-w\| \leq r$. *If it happens, particularly, the points that belong to the mesh are covered by at least one sphere, so we have*

$$\frac{\aleph_0(\underline{Pell}(E(c,R) \cap M(d)))}{\aleph_0(M(d))} = \frac{\aleph_0(\underline{Pell}(E(c,R) \cap M(d)))}{\aleph_0(\underline{Pell}(E(c,R) \cap M(d)))} = 1.$$

Given $v \in E(c,R), \exists\, w \in M(d)$ *so that* $\|v-w\| \leq d\sqrt{3}$. *If* $IP(\underline{Pell}_d(E(c,R))) = 1$, $\exists\, w_i \in \underline{Pell}_d(E(c,R))$ *so that* $\|w - w_i\| \leq d(i) - d\sqrt{3}$. *So we have*

$$\|v - w_i\| \leq \|v-w\| + \|w - w_i\| \leq d\sqrt{3} + r_i - d\sqrt{3}.$$

So there exists $w_i \in c$ *and* $r_i \in R$ *so that* $\|v - w_i\| \leq r_i$.

At this point of the work we are implementing a GRASP metaheuristic and after finishing this, we intend to use CPLEX in order to solve this problem and compare results.

The reduced number of references is due to the lack of published works about this problem in the literature.

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