Continuous Ant Colony System to Fuzzy Optimization Problems

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RESUMO

A proposta heurística de algoritmos baseados em colônia de formigas (*Ant System – AS)* foi desenvolvida por Marco Dorigo para resolver problemas de otimização combinatória, como o Problema do Caixeiro Viajante. Este algoritmo foi também adaptado por Seid H. Pourtakdoust e Hadi Nobahari para otimização de problemas contínuos (*Continuous Ant Colony Optimization Systems – CACS*). Neste trabalho, uma implementação de CACS foi utilizada para problemas de otimização não-linear com coeficientes representados por números fuzzy. Para trabalhar com esta característica, modelamos os números fuzzy através de funções de pertinência triangulares simétricas, Medida de Possibilidade, baseado em Didier Dubois e Henri Prade, para a comparação entre funções com valores fuzzy e o método de defuzzificação centróide é aplicado para obter um valor ordinário da função fuzzy no cálculo do feromônio. Nove funções testes foram usadas obtendo bons resultados – considerando a natureza imprecisa do problema (entre a solução fuzzy ótima e a solução real ordinária).
PALAVRAS CHAVES: Sistema de Colônia de Formigas, Otimização, Teoria Fuzzy.

ABSTRACT

Heuristic algorithms based in ant colonies (named ant system – AS for short) were developed by Marco Dorigo to solve combinatorial optimization problems as the traveling salesman problem. This class of algorithms was also adapted by Seid H. Pourtakdoust and Hadi Nobahari for continuous optimization problems (Continuous Ant Colony Optimization Systems – CACS). In this work, an implementation of CACS was used for nonlinear continuous optimization problems with coefficients represented by fuzzy numbers. The fuzzy numbers are modelled through symmetric triangular membership functions, Possibility Measure – based on Didier Dubois and Henri Prade's work for comparison of functions with fuzzy values – and centroid defuzzification methods to obtain the ordinary value from function values in the pheromone evaluation step. Experiments with nine benchmark functions show a good agreement – considering the imprecise nature of the problem – between the fuzzy optima and their real counterparts.
KEYWORDS: Ant Colony System, Optimization, Fuzzy Theory.

1. Introduction

Ant Colony System was developed based in the *Traveling Salesman Problem*. The ideas behind the system make it suitable for high complexity combinatorial optimization problems demanding discrete solutions. The heuristic algorithm inspired in ant colonies was developed by M. Dorigo and colleagues as we can see in Dorigo (1997) and Dorigo (1996), for example.

An extension of this algorithm for continuous function optimization was proposed by several authors, as in Socha (2008), Hu (2010) and Hu (2008), among others. The work of S.H. Pourtakdoust and H. Nobahari as in Pourtakdoust (2004) and Nobahari (2005) is an example of such an extension, with the added bonus of a simpler structure for the application of fuzzy parameters on its formulation.

The purpose of this work is the introduction of fuzzy parameters into an Ant Colony System heuristic applied to Fuzzy Mathematical Programming. The fuzzy parameters are treated as fuzzy numbers (see Kaufmann (1991)) with a double intent here: (i) to model the fuzzy parameters and (ii) make the algebraic operations fuzzy. Of course other changes are required in order to accommodate the fuzzy numbers, namely a convenient comparison operation between fuzzy quantities, an approach to evaluate the fuzzy function through ranking presented in Dubois (1986) and Dubois (1983), and finally a defuzzification process, as in Wang (2009).

The results are satisfactory, showing that Continuous Ant Colony Systems can be a valid alternative to treat Fuzzy Mathematical Programming problems.

Preliminaries

In this section, we explain some topics of Fuzzy Theory used in this work.

Definition: A fuzzy set $\tilde{c}$ on $\mathbb{R}$ is a fuzzy number, if its *membership function* is defined as follows:

$$\mu_{\tilde{c}}(x) = \begin{cases} \dfrac{x-\underline{c}}{c-\underline{c}} & \text{if } x \in [\underline{c}, c] \\ \dfrac{\overline{c}-x}{\overline{c}-c} & \text{if } x \in [c, \overline{c}] \\ 0 & \text{if } x \leq \underline{c} \text{ ou } x \geq \overline{c} \end{cases} \tag{1}$$

where $\mu_{\tilde{c}}(x): \mathbb{R} \longrightarrow [0,1]$, c is the modal value, that is, $\mu_{\tilde{c}}(c) = 1$, $\underline{c}$ and $\overline{c}$ are the inferior and superior limits, respectively $\left(\mu_{\tilde{c}}(\underline{c}) = \mu_{\tilde{c}}(\overline{c}) = 0\right)$.

We suppose that $f_{\tilde{c}}^{L}: [\underline{c}, c] \longrightarrow [0,1]$ and $f_{\tilde{c}}^{R}: [c, \overline{c}] \longrightarrow [0,1]$ are two continuous mappings from the real line $\mathbb{R}$ to the closed interval $[0,1]$ The former is a strictly increasing function (denoted by *L* – lower) and the later is a monotonically decreasing function (denoted by *R* – right). In this case, we assume that the fuzzy number is represented by a triangular function, that is $\tilde{c} = \left(c, \underline{c}, \overline{c}\right)$. Figure 1(a) represents the equation (1).

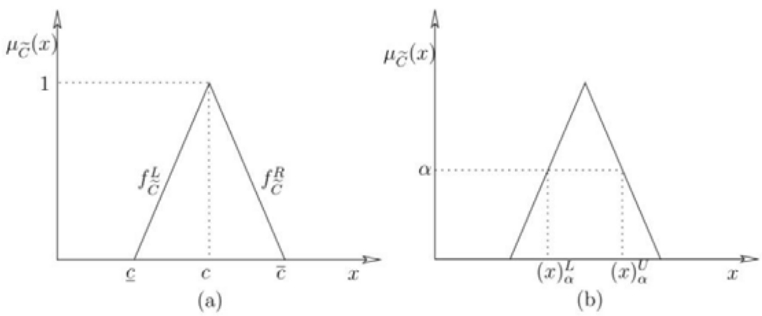

Figure 1: Membership function of triangular fuzzy numbers and α-levels representation.

In order to facilitate the operations with fuzzy numbers, we assume that an exact membership function can be approximated by using piecewise linear functions based on α-level sets.

Definition: Wang (2009) Let $\tilde{c}$ be a fuzzy number. Its α -level set $\tilde{c}_\alpha$ or α-cuts are define as

$$\begin{aligned} \tilde{c}_\alpha &= \{x \in \mathbb{R} \mid \mu_{\tilde{c}}(x) \geq \alpha\} \\ &= [\min\{x \in \mathbb{R} \mid \mu_{\tilde{c}}(x) \geq \alpha\}, \max\{x \in \mathbb{R} \mid \mu_{\tilde{c}}(x) \geq \alpha\}] \\ &= [(x)_\alpha^L, (x)_\alpha^R] \end{aligned} \tag{2}$$

Figure 1(b) illustrates equation (2).

According to Representation Theorem of Zadeh (see Pedrycz (1998)), the fuzzy number $\tilde{c}$ can also be expressed as

$$\tilde{c} = \bigcup_{\alpha \in [0,1]} \alpha\, \tilde{c}_\alpha \tag{3}$$

The α-levels representation is used to operate with fuzzy numbers, as shown in Kaufmann (1991); all other operations follow the structure presented in this reference. They are also useful in the estimation of a representative ordinary number – a process known as defuzzification. In this particular setting we used the centroid defuzzification method, as in Wang (2009). This method defines the centroid of $\tilde{c}$ as the x-axis value of the centroid as its defuzzification value, which can be expressed as:

$$D(\tilde{c}) = \frac{\int_{\underline{c}}^{\overline{c}} x\, \mu_{\tilde{c}}(x)\, dx}{\int_{\underline{c}}^{\overline{c}} \mu_{\tilde{c}}(x)\, dx} \tag{4}$$

Where

$$\begin{aligned} \int_{\underline{c}}^{\overline{c}} \mu_{\tilde{c}}(x)\, dx &= \frac{1}{2n}\left[\left((x)_{\alpha_0}^U - (x)_{\alpha_0}^L\right) + 2\sum_{i=1}^{n-1}\left((x)_{\alpha_i}^U - (x)_{\alpha_i}^L\right)\right] \\ \int_{\underline{c}}^{\overline{c}} x\mu_{\tilde{c}}(x)\, dx &= \frac{1}{6n}\left[\left((x)_{\alpha_0}^{2U} - (x)_{\alpha_0}^{2L}\right) + 2\sum_{i=1}^{n-1}\left((x)_{\alpha_i}^{2U} - (x)_{\alpha_i}^{2L}\right) \right. \\ &\quad \left. + \sum_{i=1}^{n-1}\left((x)_{\alpha_i}^U.(x)_{\alpha_{i+1}}^U - (x)_{\alpha_i}^L.(x)_{\alpha_{i+1}}^L\right)\right] \end{aligned} \tag{5}$$

In order to compare (or rank) fuzzy numbers, $\tilde{c}_1$ and $\tilde{c}_2$ for instance, we can apply a comparison measure built upon the Possibility Measure, as presented in Dubois (1986) and Dubois (1983). In this context, if we want to decide whether $\tilde{c}_1 > \tilde{c}_2$ or not, we use the following measure – remembering that $\tilde{c}_1 = (c_1, \underline{c}_1, \overline{c}_1)$ and $\tilde{c}_2 = (c_2, \underline{c}_2, \overline{c}_2)$:

$$\mathrm{Poss}(\tilde{c}_1 \geq \tilde{c}_2)\max\left(0, \min\left(1, 1 + \frac{c_1 - c_2}{(c_1 - \underline{c}_1) + (\overline{c}_2 - c_2)}\right)\right)$$

$$\mathrm{Poss}(\tilde{c}_1 > \tilde{c}_2)\max\left(0, \min\left(1, 1 + \frac{c_1 - c_2 + (c_1 - \underline{c}_1)}{(c_1 - \underline{c}_1) + (\overline{c}_2 - c_2)}\right)\right)$$

The first equation above will be denoted by PSE, which stands for *exceedance possibility* and the second one by PS, for *strict exceedance possibility*. According to Dubois (1986), these formulas hold except when the sums of the spreads in the denominators are zero, which occurs when $\tilde{c}_1$ and $\tilde{c}_2$ are ordinary numbers.

So we assume that $\tilde{c}_1 > \tilde{c}_2$ when:

$$\mathrm{Poss}(\tilde{c}_1 > \tilde{c}_2) > \alpha, \quad \alpha \in]0,1] \tag{7}$$

and

$$\min[\mathrm{Poss}(\tilde{c}_1 \geq \tilde{c}_2), \mathrm{Poss}(\tilde{c}_2 \geq \tilde{c}_1)] < 1 \tag{8}$$

Condition (8) guarantees that $\tilde{c}_1 \neq \tilde{c}_2$.

The topics presented in this section were directly used in the computational implementation, so their descriptions are brief and without details about the theoretical foundations.

The problem

As described in Jamison (1999), a fuzzy function is used when some data about the problem is not precisely known. Here a function with fuzzy parameters can be denoted by:

$$\begin{array}{ll} \min & f(\tilde{c}, x) \\ \mathrm{S.\,to} & x_i \in [a_i, b_i] \quad i = 1{:}n \\ & x \in \mathbb{R}^n \end{array} \tag{9}$$

where $x \in \mathbb{R}^n$ and $\tilde{c}$ is a vector whose entries are fuzzy numbers, such that $\tilde{c} \in \mathrm{F}(\mathbb{R})$, $\mathrm{F}(\mathbb{R})$ is a fuzzy set over $\mathbb{R}$ and $f(\tilde{c}, x)\colon \mathrm{F}(\mathbb{R})^n \longrightarrow \mathrm{F}(\mathbb{R})$. The interval $[a, b]$ is the region which the minimum value of the function, denoted by $\bar{x}$, occurs.

Even though problem (9) is an irrestrict one, we determine a search region for vector x where the nonlinear function is evaluated.

Continuous Ant Colony System (CACS) with Fuzzy Parameters

The heuristic method developed by Pourtakdoust (2004) is a modification over the heuristic *Ant Colony System* – ACS, preserving all of its major characteristics. Some important aspects are related here.

Continuous Pheromone Model

As reported in Pourtakdoust (2004) and Nobahari (2005), the pheromone deposition occurs over a continuous space. For fuzzy problem (9), this step involves only ordinary numbers because it involves only information about the vector x.

Consider a food source surrounded by several ants. The ant's aggregation around the food source causes the highest pheromone intensity to occur at the food source position. Then, increasing the distance of a sample point from the food source will decrease its pheromone concentration. This model uses a *Probability Distribution Function* (PDF), which determines the probability of choosing each point x within the interval $[a, b]$.

The normal PDF can be used at the state transition rule since the center of which is the last best global solution and its variance depends on the aggregation of the promising areas around the best one, so it contains exploitation behavior. In the other hand, a normal PDF permits all points of the search space to be chosen, either close to or far from the current solution, so it also contains exploration behavior.

Pheromone Update

At the start of the algorithm, there is no information available about the minimum point and the ants chose their destination only by exploration.

During each iteration pheromone distribution over the search space will be updated using the acquired knowledge of the evaluated points by the ants. This process gradually increases the exploitation behavior of the algorithm, while its exploration behavior will decrease, that is, the value of objective function is evaluated for the new selected points by the ants. Then, the best point found from the beginning of the trial is assigned to $x_{\min}$. Also the value of σ is update based on the evaluated points during the last iteration and the aggregation of those points around $x_{\min}$. Then a concept of weighted variance is defined as follows:

$$\sigma^2 = \frac{\sum_{j=1}^{k}\left(\frac{1}{D(f_j)-D(f_{\min})} - (x_j - x_{\min})\right)}{\sum_{j=1}^{k}\left(\frac{1}{D(f_j)-D(f_{\min})}\right)} \quad (10)$$

for all j in which $D(f_j) \neq D(f_{\min})$, $D(\cdot)$ meaning the fuzzy number defuzzification and k is the number of ants. This strategy means that the center of the region discovered during the subsequent iterations is the last best point and the narrowness of its width is dependent on the aggregation of the other competitors around the best one. The closer the better solutions go to the best one, the smaller σ is assigned to the next iteration.

Algorithm

The algorithm description, based in Nobahari (2005), is shown below, including the modification for the fuzzy problem (9).

Step 1 choose randomly the initially guessed minimum point $\bar{x}_{\min}$ over the space and calculate the value of the function $f(\tilde{c},\ \bar{x}) = f_{\min}$, calculate $D(f_{\min})$ using (4). For each x_i use a uniform PDF over the interval $[a_i, b_i]$.

Step 2 Set the initial value of weighted variance for each pheromone intensity distribution function $\sigma_i = 3(b_i - a_i)$, $i = 1{:}n$. It will be large enough to approximately generate uniformly distributed initial values of x_i within the interval $[a_i, b_i]$.

Step 3 Send ants to points $(x_1, x_2, \cdots, x_n)_j$, $j = 1{:}k$. To generate these random locations, a random generator with normal PDF is utilized for each x_i, where its mean and variance are $(x_j)_{\min}$ and σ_i respectively. If x_i is outside the given interval $[a_i, b_i]$, it is discarded and a new x_i is generated again.

Step 4 Evaluate f, at each discovered point, namely $f_1, f_2, \cdots, f_k$. Determine the minimum f_m and compare these value using (7) and (8) with the current minimum value $f_{\min}$ and determine the updated $f_{\min}$ and its associated $(x_1,\ x_2, \cdots, x_n)_{\min}$.

Step 5 If a stopping criterion is satisfied (usually the number of iterations) then **stop**, else update the weighted variance parameter σ_i for each variable x_i using (10); go back to **Step 3**.

Experiments

The following experiments employ test functions from Poutakdoust (2004) and one function from Vicente (2006). Also, each function will be presented on its own table, together the original result from their respective references.

All fuzzy numbers have 10% of uncertainty. They are represented as $[c, \bar{c}, \underline{c}]$, where c is the modal value of the number $(\mu_{\tilde{c}}(x) = 1)$, $\bar{c}$ and $\underline{c}$ are the inferior and superior values for the number $(\mu_{\tilde{c}}(x) = 0)$, respectively.

Each table is composed by four columns, with the first one being the number of ants, followed by the best crisp result obtained on the cited reference, then the fuzzy objective function value and finally, on the last column, the defuzzified objective function number. The table was constructed upon an implementation using *Scilab version 5.3.0*.

Function 1

$$f_1(\tilde{c}, x) = \widetilde{3905.93} - \widetilde{10}(x_1^2 - x_2)^2 - (1 - x_1)^2,$$
$$-2.048 \leq x_1, x_2 \leq 2.048$$

k	de Vicente (2006)	$f_1(\tilde{c},x)$	$D(f_1)$
100	-3905.93	[-3905.93, -3944.99, -3866.87]	-3905.93
200		[-3905.93, -3944.99, -3866.87]	-3905.93
500		[-3905.93, -3944.99, -3866.87]	-3905.93

Table 1: Results for function f_1.

Function 2

$$f_2(\tilde{c},x) = \widetilde{100}(x_1^2 - x_2)^2 + (1-x_1)^2,$$
$$-2.05 \leq x_1, x_2 \leq 2.05$$

k	Pourtakdoust (2004)	$f_2(\tilde{c},x)$	$D(f_2)$
100	1.6e-33	[1.9036e-19, -0.01, 0.01]	0
200	3.2e-22	[2.743e-21, -0.01, 0.01]	0
500	1.7e-12	[2.299e-20, -0.01, 0.01]	0

Table 2: Results for function f_2.

Function 3

$$f_3(\tilde{c},x) = \tilde{1}x_1^2 + \tilde{1}x_2^2 + \tilde{1}x_3^2, \; -5.12 \leq x_1, x_2, x_3 \leq 5.12$$

k	Pourtakdoust (2004)	$f_3(\tilde{c},x)$	$D(f_3)$
100	3.3e-37	[1.394e-34, 1.255e-34, 1.534e-34]	1.394e-34
200	1.5e-20	[9.094e-36, 8.184e-36, 1e-35]	9.094e-36
500	3.0e-09	[3.761e-35, 3.385e-35, 4.137e-25]	3.761e-35

Table 3: Results for function f_3.

Function 4

$$f_4(\tilde{c},x) = \widetilde{0.5} + \frac{\sin^2(x_1^2 + x_2^2)^{1/2}}{\tilde{1} + \widetilde{0.001}(x_1^2 + x_2^2)}, \; -100 \leq x_1, x_2 \leq 100$$

k	Pourtakdoust (2004)	$f_4(\tilde{c},x)$	$D(f_4)$
100	7.8e-3	[0, -0.1611, 0.1409]	-0.005056
200	7.7e-3	[0, -0.1611, 0.1409]	-0.005056
500	1.4e-2	[0, -0.1611, 0.1409]	-0.005056

Table 4: Results for function f_4.

Funcion 5

$$f_5(\tilde{c},x) = \widetilde{50} + \sum_{i=1}^{5}\left(x_i^2 - \widetilde{10}\cos(\tilde{2}\tilde{\pi}x_i)\right), \; -5.12 \leq x_i \leq 5.12$$

k	Pourtakdoust (2004)	$f_5(\tilde{c},x)$	$D(f_5)$
100	4.9	[0, -10, 10]	1.108e-14
200	7.1	[0, -10, 10]	1.108e-14
500	9.4	[0, -10, 10]	1.108e-14

Table 5: Results for function f_5.

Function 6

$$f_6(\tilde{c},x) = \tilde{1} + \sum_{i=1}^{2} \frac{x_i^2}{\widetilde{4000}} - \prod_{i=1}^{2} \cos\left(\frac{x_i}{\sqrt{i}}\right), \quad -5.12 \le x_i \le 5.12$$

The number 4000 has only 1% of fuzzy uncertainty.

k	Pourtakdoust (2004)	$f_6(\tilde{c},x)$	$D(f_6)$
100	4.1e-3	[-9.375e-8, -1.000001, -0.0999999]	-9.376e-8
200	2.7e-3	[-9.375e-8, -1.000001, -0.0999999]	-9.376e-8
500	1.1e-3	[-9.375e-8, -1.000001, -0.0999999]	-9.376e-8

Table 6: Results for function f_6.

Function 7

$$f_7(\tilde{c},x) = \tilde{1} + \sum_{i=1}^{5} \frac{x_i^2}{\widetilde{4000}} - \prod_{i=1}^{5} \cos\left(\frac{x_i}{\sqrt{i}}\right), \quad -5.12 \le x_i \le 5.12$$

The number 4000 has only 1% of fuzzy uncertainty.

k	Pourtakdoust (2004)	$f_7(\tilde{c},x)$	$D(f_7)$
100	7.8e-3	[-9.375e-8, -1.000001, -0.0999999]	-9.376e-8
200	7.7e-3	[-9.375e-8, -1.000001, -0.0999999]	-9.376e-8
500	1.4e-2	[-9.375e-8, -1.000001, -0.0999999]	-9.376e-8

Table 7: Results for function f_7.

Function 8

$$f_8(\tilde{c},x) = (x_1^2 + x_2^2)^{0.25}\left(\tilde{1} + \sin^2\left(\widetilde{50}(x_1^2 + x_2^2)^{0.1}\right)\right),$$
$$-100 \le x_i \le 100$$

k	Pourtakdoust (2004)	$f_8(\tilde{c},x)$	$D(f_8)$
100	2.5e-3	[2.176e-2, 1.935e-2, 1.176e-2]	0.0176
200	5.9e-2	[3.509e-3, 1.815e-3, 2.219e-3]	0.00202
500	3.8e-1	[4.228e-2, 1.945e-2, 2.377e-2]	0.02168

Table 8: Results for function f_8.

Function 9

$$f_9(\tilde{c},x) = -\widetilde{20}\exp\left(-\widetilde{0.2}\sqrt{\frac{1}{n}\sum_{i=1}^{30} x_i^2}\right) - \exp\left(\frac{1}{n}\sum_{i=1}^{30}\cos(\tilde{2}\tilde{\pi}x_i)\right) + \widetilde{20} + \tilde{e},$$
$$-32 \le x_i \le 32$$

k	$f_9(\tilde{c},x)$	$D(f_9)$
100	[1.421e-14, -4.27, 4.27]	5.547e-17
200	[1.421e-14, -4.27, 4.27]	5.547e-17
500	[1.421e-14, -4.27, 4.27]	5.547e-17

Table 9: Results for function f_9.Optimum crisp solution $f_9(\mathbf{0}) = 0$ (see Yao (1999)).

Conclusions

Fuzzy theory was proposed by L. A. Zadeh in 1965, as a tool to help the quantification of inherent imprecisions on the subject being studied. This theory quickly spread to several different fields, ranging from Engineering, Informatics and Mathematics to medical diagnosis, plague control e so on.

Mathematical Programming is by itself a very important decision tool in several application areas. Aggregation of Fuzzy characteristics to it allows the modelling of uncertainties when the available data is not exactly know or has inherent inaccuracies, making Mathematical Programming an even more powerful resource.

In this sense, the quest for optimization methods that successfully embraces Fuzzy Theory has been the focus of this work. Here, we have introduced uncertainty in the coefficients of the objective function. These uncertainties were modelled by fuzzy numbers. Its application on the Continuous Ant Colony System heuristics required some adaptations, as outlined here.

We tested 9 functions from the literature, with 77% of them giving equivalent or better results when compared with their crisp counterparts. Even the 23% of the test cases that performed worse, were kept inside de viable solution space and also relatively close to the crisp solution.

So, despite the fact that this implementation still needs some improvements – for instance, incorporating different ant colony heuristics, as proposed in Socha (2008) and Hu (2010) – its results are compatible with the ordinary ones, allowing a flexibilization in the modelling of real case, where crisp Mathematical Programming is not directly applicable.

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