# FUZZY DUAL PROGRAMMING: AN INTRODUCTION

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## Resumo

Considerando problemas de otimização onde parâmetros e variáveis apresentam incerteza, este trabalho introduz os conceitos de números, vetores e matrizes fuzzy duais assim como as operações que constituem o cálculo fuzzy dual aplicado à variáveis fuzzy. Se propõe então a formulação de problemas de otimização de acordo com este novo formalismo mostrando que a cada problema fuzzy dual é associado um conjunto finito de problemas clássicos de programação matemática incluído o caso em que o conjunto admissivel é formulado usando restrições LMI de tipo fuzzy duais.

**PALAVARAS CHAVE. Variáveis fuzzy, números duais, otimização.**

**Área principal: Programação matemática.**

## Abstract

To tackle optimization problems with uncertain parameters and variables, this communication introduces the concepts of fuzzy dual numbers, vectors and matrices and considers basic operations which compose fuzzy dual calculus applied to fuzzy quantities. The formulation of optimization problems using this new formalism is discussed. It is shown that each fuzzy dual programming problem generates a finite set of classical optimization problems, even in the case in which the feasible set is defined using fuzzy dual LMI constraints.

**KEYWORDS. Fuzzy variables, dual numbers, optimization.**

**Main area: Mathematical programming**

## 1. Introduction

While deterministic optimization problems are formulated with known parameters, very often real world problems include unknown parameters (Delgado et al., 1987). When the parameters are only known to remain within given bounds, one way to tackle such problems is through robust optimization (Ben-Tal et al., 2009). When probability distributions are available for their values, stochastic optimization techniques (Ruszczynski et al., 2003) may provide the most expected feasible solution. An intermediate approach adopting the fuzzy formalism to represent the parameter uncertainties, has been also developed (Zimmermann, 1986). These three approaches lead in general to cumbersome computations. Also, in many situations the optimal solution cannot be applied exactly according to implementation constraints which have not been considered explicitly in the formulation of the problem. In that case post optimization sensibility analysis (Gal et al., 1997) resulting often in an important computational effort must be performed.

In this communication, a new formalism is proposed to treat parameter uncertainty and solution diversion in mathematical optimization problems. This formalism adopts a simplified version of fuzzy numbers and adopts some elements of dual numbers calculus. Dual numbers were introduced by Clifford in the nineteenth century as part of the "Theory of engines" which was based on the use of a nilpotent operator noted $\varepsilon$. The use of dual numbers to represent feasible kinematics for rigid articulated bodies was proposed by Kotelnikov of the University of Kazan at the end of the nineteenth century. More recently several authors (Cheng 1994, Penestrelli et al., 2007) have developed computer tools for dual numbers calculus. The proposed special class of numbers, dual fuzzy numbers (Cosenza et al., 2011), (Mora-Camino et al. 2012), integrates such a nilpotent operator and is representative of symmetrical fuzzy numbers. Here it is shown how to tackle the uncertainty on the value of parameters or variables involved in the formulation and solution of decision problems with fuzzy dual numbers, vectors and matrices. Fuzzy dual equalities and inequalities as well as fuzzy dual LMI constraints are considered before introducing fuzzy dual programming with real and fuzzy dual variables.

## 2. Fuzzy dual numbers, vectors and matrices and fuzzy dual calculus

The set of *fuzzy dual numbers* is the set $\tilde{\Delta}$ of the dual numbers of the form $a+\varepsilon b$ such as $a \in R, b \in R^+$ where $a$ is the primal part and $b$ is the dual part of the fuzzy dual number.

Observe that a crisp fuzzy dual number will be such as $b$ is equal to zero then it loses both its dual and its fuzzy attributes. To each fuzzy dual number is attached a fuzzy symmetrical number whose graphical representation is given below where μ is a symmetrical membership function defined over $R$:

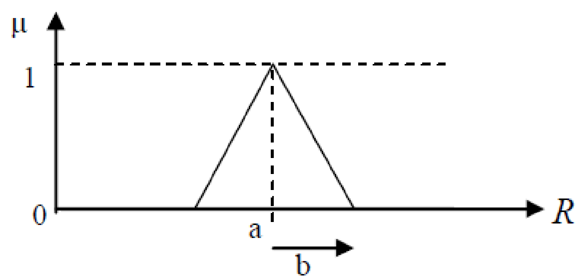


Figure 1. Example of graphical representation of a triangular fuzzy dual number

The fuzzy dual addition of fuzzy dual numbers, written $\tilde{+}$, is identical to that of dual numbers and is given by:

$$(x_1+\varepsilon\, y_1)\,\tilde{+}\,(x_2+\varepsilon\, y_2)=(x_1+x_2)+\varepsilon\,(y_1+y_2) \tag{1}$$

Its neutral element is $(0+0\varepsilon)$, written $\tilde{0}$. The fuzzy dual product of two fuzzy dual numbers, written $\tilde{\bullet}$, is given by:

$$(x_1+\varepsilon\, y_1)\,\tilde{\bullet}\,(x_2+\varepsilon\, y_2)=(x_1.x_2+\varepsilon(|x_1|\cdot y_2+|x_2|\cdot y_1)) \tag{2}$$

The fuzzy dual product has been chosen in that way to preserve the fuzzy interpretation of the dual part of the fuzzy dual numbers but it makes a difference with classical dual calculus. The neutral element of fuzzy dual multiplication is $(1+0\varepsilon)$, written $\tilde{1}$ and only non zero crisp numbers have an inverse. Both internal operations, fuzzy dual addition and fuzzy dual multiplication, are commutative and associative, while the fuzzy dual multiplication is distributive with respect to the fuzzy dual addition. Observe that the nilpotent property of operator $\varepsilon$ is maintained: $\varepsilon\,\tilde{\bullet}\,\varepsilon=\varepsilon^2=\tilde{0}$. It appears also that fuzzy dual calculus is quite simpler than common fuzzy calculus (Kosinsky 2006, Nasseri 2006)).

Let $E$ be an Euclidean space of dimension $p$ over $R$, we construct a set $\tilde{E}$ composed of pairs of vectors which are called *fuzzy dual vectors* taken from the Cartesian product $E\times E^+$, where $E^+$ is the positive half-space of $E$ in its canonical basis. The following operations are defined over $\tilde{E}$:

- Addition:

$$(a,b)+(c,d)=(a+c,b+d) \qquad \forall\, a,b\in E \quad \forall c,d\in E^+ \tag{3}$$

- Multiplication by a fuzzy dual scalar $\lambda+\varepsilon\,\mu$:

$$(\lambda+\varepsilon\,\mu)\,(a,b)=(\lambda\, a,|\lambda|\, b+\mu|a|) \quad \forall\lambda+\varepsilon\,\mu\in\tilde{\Delta},\forall(a,b)\in\tilde{E} \tag{4}$$

We can write then:

$$(a,b)=a+\varepsilon\, b \qquad \forall(a,b)\in\tilde{E} \tag{5}$$

where the real and dual part of the fuzzy dual vector $a+\varepsilon\, b$ de $\tilde{E}$ are:

$$r(a+\varepsilon\, b))=a \qquad d(a+\varepsilon\, b))=b \tag{6}$$

A pseudo scalar product is defined by:

$$u*v=r(u).r(v)+\varepsilon\,(|r(u)|.d(v)+d(u).|r(v)|) \quad \forall\, u,v\in\tilde{E} \tag{7}$$

where "*" represents the inner product in $\tilde{E}$ and "." represents the inner product in $E$. Two fuzzy dual vectors $u$ and $v$ are said to be orthogonal if $u*v=\tilde{0}$ where $\tilde{0}$ is the neutral element for the addition of fuzzy dual vectors.

Now we introduce the set $\tilde{M}_n$ of *fuzzy dual square matrices* of order $n\times n$ following the same idea used to build fuzzy dual numbers and fuzzy dual vectors.

Then a fuzzy dual matrix $A$ is be such as:

$$A=[a_{ij}]=[r(a_{ij})+\varepsilon\, d(a_{ij})]=r(A)+\varepsilon\, d(A) \tag{8}$$

where $r(A)$ is a $R^{n\times n}$ matrix and $d(A)$ is a positive $R^{n\times n}$ matrix. Basic operations over dual square matrices are such that:

$$A+B=r(A)+r(B)+\varepsilon\,(d(A)+d(B)) \qquad \forall A,B\in\tilde{M} \tag{9.1}$$

$$A\bullet B=r(A)\, r(B)+\varepsilon\,(|r(A)|\cdot d(B)+d(A)|r(B)|) \quad \forall A,B\in\tilde{M} \tag{9.2}$$

$$\lambda\ A = r(\lambda)\ r(A) + \varepsilon\ (|r(\lambda)|\ r(A) + d(\lambda)\ |r(A)|) \quad \forall \lambda \in \tilde{\Delta}, \forall A \in \tilde{M}_3 \tag{9.3}$$

The product of a fuzzy dual square matrix by a fuzzy dual vector $u$ is here a fuzzy dual vector such as:

$$A \times u = r(A) r(u) + \varepsilon\ (|r(A)|\ d(u) + d(A)\ |r(u)|) \tag{10}$$

### 3. Pseudo norm for fuzzy dual numbers and vectors

The pseudo ($\tilde{\Delta}$ is not a vector space) norm of a dual fuzzy number is given by:

$$\|a + \varepsilon\, b\| = |a| + \rho\, b \ \in R^+ \tag{11}$$

where $\rho > 0$ is a shape parameter. The shape parameter can be defined as:

$$\rho = \frac{1}{2b} \int_{y \in R^+} \mu(y)\ dy \tag{12}$$

Figure 2 displays standard fuzzy symmetrical numbers with different shape parameters.

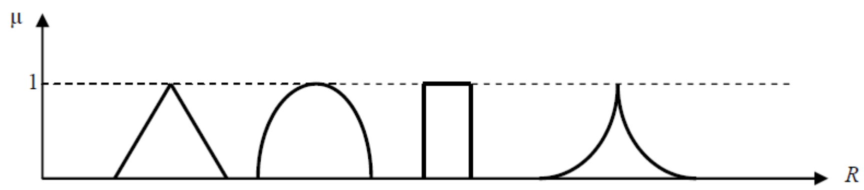


Figure 2. Examples of shapes for fuzzy dual numbers

The present study is restricted to the triangular case where $\rho = 1/2$.

The following properties are met by this pseudo norm whatever the values of the shape parameters :

$$\forall\ a + \varepsilon\, b \in \tilde{\Delta} : \|a + \varepsilon\, b\| \geq 0 \tag{13.1}$$

$$\forall a \in R, \forall b \in R^+ \quad \|a + \varepsilon\ b\| = 0 \Rightarrow a = b = 0 \tag{13.2}$$

$$\|(a + \varepsilon b) + (\alpha + \varepsilon \beta)\| \leq \|a + \varepsilon b\| + \|\alpha + \varepsilon \beta\| \quad \forall a, \alpha \in R, \forall b, \beta \in R^+ \tag{13.3}$$

$$\|\lambda (a + \varepsilon b)\| = \lambda \|a + \varepsilon b\| \quad \forall a \in R, \forall b, \lambda \in R^+ \tag{13.4}$$

For a dual vector $u$ of $\tilde{E}$ with $r(u) \neq 0$, we write:

$$\|u\|_D = \|r(u)\| + \rho\ |r(u)| \cdot d(u) / \|r(u)\| \tag{14}$$

where $\| \ \|$ is the Euclidean norm associated to $E$ which has a vector space structure. If $u = \tilde{0}$ we take $\|u\|_D = \tilde{0}$ and we speak of pseudo-fuzzy dual norm while an orthonormal basis can be considered in $\tilde{E}$ .

## 4. Fuzzy dual equalities and inequalities

Partial orders between fuzzy dual numbers can be introduced using the above pseudo norm. First a strong partial order written $\breve{\geq}$ can be defined over $\tilde{\Delta}$ by:

$$\forall\, a_1 + \varepsilon b_1 , a_2 + \varepsilon b_2 \in \tilde{\Delta} : a_1 + \varepsilon b_1 \,\breve{\geq}\, a_2 + \varepsilon b_2 \Leftrightarrow a_1 - \rho b_1 \geq a_2 + \rho b_2 \qquad (15)$$

Then a weak partial order written $\hat{\geq}$ can be also defined over $\tilde{\Delta}$ by:

$$\forall\, a_1 + \varepsilon b_1 , a_2 + \varepsilon b_2 \in \tilde{\Delta} : \|a_1 + \varepsilon b_1\| \,\hat{\geq}\, \|(a_2 + \varepsilon b_2)\| \Leftrightarrow a_1 + \rho b_1 \geq a_2 - \rho b_2 \qquad (16)$$

Figures 3 and 4 displays different partial orders between pairs of dual fuzzy numbers and inequalities between fuzzy dual numbers are quite different from those used with classical fuzzy numbers.

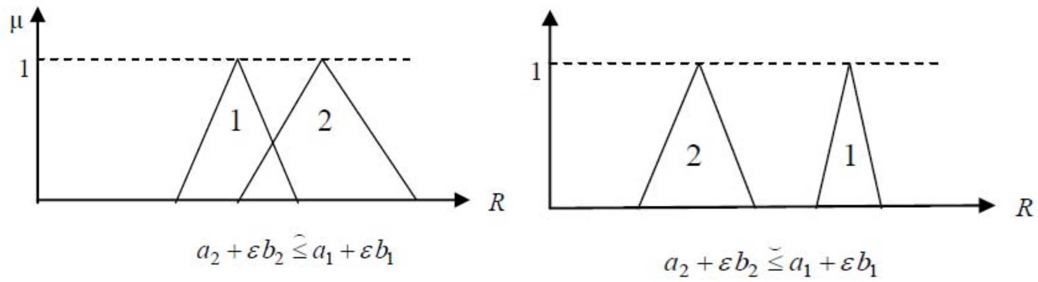


Figure 3. Examples of inequalities (weak and strong) between fuzzy dual numbers

More, a fuzzy equality written $\cong$ can be defined between two fuzzy dual numbers by:

$$\begin{aligned} &\forall\, a_1 + \varepsilon b_1 , a_2 + \varepsilon b_2 \in \tilde{\Delta} : \|a_1 + \varepsilon b_1\| \cong \|(a_2 + \varepsilon b_2)\| \\ &\Leftrightarrow a_2 \in [a_1 - \rho b_1 , a_1 + \rho b_1] \; et \; a_1 \in [a_2 - \rho b_2 , a_2 + \rho b_2] \end{aligned} \qquad (17)$$

Figures 5 and 6 display examples with different degrees of fuzzy equality between pairs of fuzzy dual numbers.

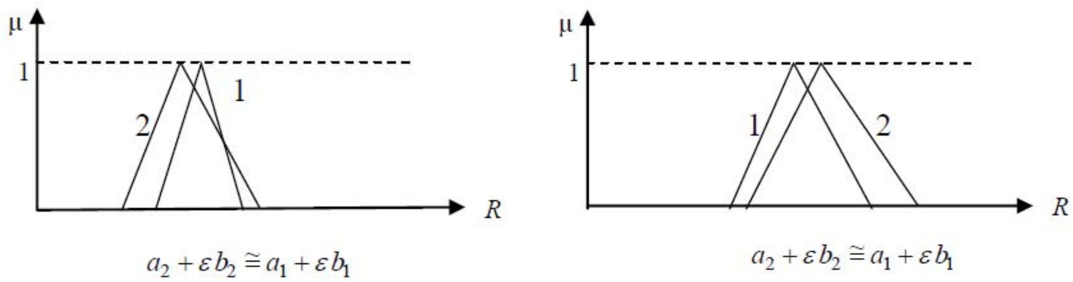


Figure 4. Examples of fuzzy equality between fuzzy dual numbers

## 5. Fuzzy dual LMI constrained sets

Let $S$ be the convex set of symmetric matrices on $R^{n\times n}$ with non-negative eigenvalues. Membership of a matrix $A$ in $S$ is written $A \geq 0$ and a linear matrix inequality-LMI (Williams et al., 2007)) is then written $M(z) \geq 0$ where $M$ is an affine map of $R^m$ into $S$ such that:

$$M(z) = M_0 + \sum_{i=1}^{m} z_i \ M_i \tag{18}$$

where the matrices $M_i, i = 0, 1, \cdots, m$ are symmetric matrices of $R^{n\times n}$. The LMI associated with the field $M$ (z) is the set given by:

$$\Lambda = \left\{ \underline{z} \in R^m : \forall \underline{x} \in R^n \quad \underline{x}' M(\underline{z})\underline{x} \ \geq 0 \right\} \tag{19}$$

LMI's are convex closed domains and many convex sets can be represented using this formalism (Boyd 1994). Here we extend this concept to consider fuzzy dual LMI's.
Let $\tilde{M}$ be a fuzzy dual matrix of $R^{n\times n}$ parameterized by $\underline{z} \in R^m$ and such as:

$$\tilde{M}(z) = M_0 + \sum_{i=1}^{m} z_i \ M_i = (M_0^R + \varepsilon M_0^D) + \sum_{i=1}^{m} z_i (M_i^R + \varepsilon M_i^D) \tag{20}$$

where $M_i^R, i \in \{0,1,\cdots,m\}$ are symmetric matrices with real coefficients and $M_i^D, i \in \{0,1,\cdots,m\}$ are symmetric matrices with positive coefficients and $\tilde{M}(z)$ can be written as:

$$\tilde{M}(z) = (M_0^R + \sum_{i=1}^{m} z_i M_i^R) + \varepsilon (M_0^D + \sum_{i=1}^{m} z_i \ M_i^D) \tag{21}$$

This matrix is dual fuzzy if $\forall \underline{z} \in R^m$ it is such that:

$$M_0^D + \sum_{i=1}^{m} z_i \ M_i^D \geq O \tag{22}$$

where $O$ is the zero matrix of dimensions $n \times n$.
The LMI-hard domain associated with the fuzzy dual matrix $\tilde{M}$ is then the domain $\Lambda_F$ of $R^m$ defined by:

$$\Lambda_F = \left\{ \underline{z} \in R^m : \forall \underline{x} \in R^n \quad \underline{x}' M(z)\underline{x} \breve{\geq} \tilde{0} \right\} \tag{23}$$

The condition $x' \tilde{M}(z) \, x \breve{\geq} 0$ can also be written:

$$\underline{x}'(M_0^R + \sum_{i=1}^{m} z_i M_i^R) + \varepsilon (M_0^D + \sum_{i=1}^{m} z_i \ M_i^D) \underline{x} \breve{\geq} \tilde{0} \tag{24}$$

or

$$\underline{x}'(M_0^R + \sum_{i=1}^{m} z_i M_i^R)\underline{x} - \rho \ \underline{x}'(M_0^D + \sum_{i=1}^{m} z_i \ M_i^D)\underline{x} \geq 0 \tag{25}$$

and finally:

$$\underline{x}'((M_0^R - \rho M_0^D) + \sum_{i=1}^{m} z_i (M_i^R - \rho M_i^D))\underline{x} \geq 0 \tag{26}$$

The LMI domain $\Lambda_F$ associated with the matrix $M$ in $R^m$ is such that: $\underline{z} \in R^m$ with

$$\begin{cases} M_0^D + \sum_{i=1}^{m} z_i M_i^D \geq O \\ (M_0^R - \rho M_0^D) + \sum_{i=1}^{m} z_i (M_i^R - \rho M_i^D) \in SDP_n \end{cases} \tag{27}$$

Here again we see that it is a convex domain. If we consider the LMI domains $\Lambda_R$ and $\Lambda_D$ associated respectively with the certain and uncertain matrices:

$$M_0^R + \sum_{i=1}^{m} z_i M_i^R \quad \text{and} \quad M_0^D + \sum_{i=1}^{m} z_i M_i^D \tag{28}$$

It is easy to verify that $\Lambda_F \subset \Lambda_R \cap \Lambda_D$ .

The LMI-weak domain associated with the fuzzy dual matrix $M$ in $R^m$ is defined by:

$$\left\{ \underline{z} \in R^m : \forall \underline{x} \in R^n \quad \underline{x}' \tilde{M}(z) \underline{x} \mathrel{\widehat{\geq}} \tilde{0} \right\} \tag{29}$$

The condition $x' M(z) x \mathrel{\breve{\geq}} 0$ can also be written:

$$\underline{x}' (M_0^R + \sum_{i=1}^{m} z_i M_i^R) + \varepsilon (M_0^D + \sum_{i=1}^{m} z_i M_i^D) \underline{x} \mathrel{\widehat{\geq}} \tilde{0} \tag{30}$$

or

$$\underline{x}' (M_0^R + \sum_{i=1}^{m} z_i M_i^R) \underline{x} + \rho \ \underline{x}' (M_0^D + \sum_{i=1}^{m} z_i M_i^D) \underline{x} \geq 0 \tag{31}$$

and finally:

$$\underline{x}' ((M_0^R + \rho M_0^D) + \sum_{i=1}^{m} z_i (M_i^R + \rho M_i^D)) \underline{x} \geq 0 \tag{32}$$

The LMI-weak domain associated with the matrix $\tilde{M}$ in $R^m$ is such that:

$$\underline{z} \in R^m \quad \text{with} \quad \begin{cases} M_0^D + \sum_{i=1}^{m} z_i M_i^D \geq O \\ (M_0^R + \rho M_0^D) + \sum_{i=1}^{m} z_i (M_i^R + \rho M_i^D) \in SDP_n \end{cases} \tag{33}$$

Here again we see that it is a convex domain. It also easy to check that $\Lambda_R \cap \Lambda_D \subset \Lambda_f$ . We can then define the fuzzy boundary of the LMI domain $\Phi$ by:

$$\Phi = \Lambda_f - \Lambda_F \tag{34}$$

For example, consider the fuzzy dual LMI domain given by:

$$\tilde{M}(\underline{z}) = \begin{bmatrix} 1 - z_1 & z_1 + z_2 \\ z_1 + z_2 & 2 - z_2 \end{bmatrix} + \varepsilon \begin{bmatrix} 0.1 & 0.1 z_2 \\ 0.1 z_2 & 0.2 z_1 \end{bmatrix} \tag{35}$$

Here the different LMI domains are such as:

$$\Lambda_D = \{ z_1 \geq 0, z_2 \geq 0 \} \tag{36.1}$$

$$\Lambda_R = \{z_1 \le 1,\, z_2 \le 2,\, (1-z_1)(2-z_2)-(z_1+z_2) \ge 0\} \tag{36.2}$$

$$\Lambda_F = \begin{Bmatrix} 0.9 \ge z_1 \ge 0,\, z_2 \ge 0,\, 2-0.2\,z_1 - z_2 \ge 0, \\ (0.9-z_1)(2-0.2z_1 - z_2)-(z_1+0.9z_2)^2 \ge 0 \end{Bmatrix} \tag{36.3}$$

$$\Lambda_f = \begin{Bmatrix} 1.1 \ge z_1 \ge 0,\, z_2 \ge 0,\, 2+0.2\,z_1 - z_2 \ge 0, \\ (1.1-z_1)(2+0.2z_1 - z_2)-(z_1+1.1z_2)^2 \ge 0 \end{Bmatrix} \tag{36.4}$$

Figure 5 shows the resulting fuzzy domain which presents on one side a sharp border ( $z_1 \ge 0, z_2 \ge 0$ ) and on the first quarter of the plane a fuzzy border.

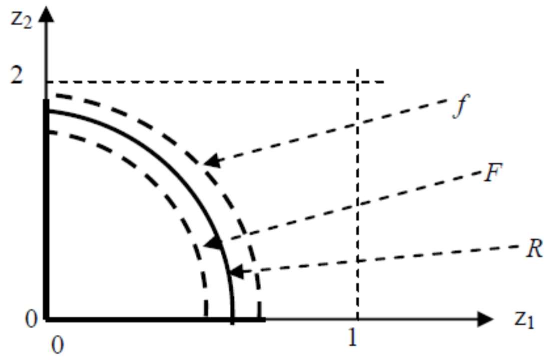


**Figure 5. Example of a fuzzy dual LMI domain**

## 6. Fuzzy dual programming with fuzzy dual parameters and real variables

It is now possible to introduce fuzzy dual formulations of uncertain mathematical programming problems. Here we will consider only the case of linear programming, but the formalism can be applied to other classes of objective functions and restrictions. Let then consider problem *D₀* which is a fuzzy dual linear programming problem with fuzzy dual constraints and real decision variables:

$$\min_{x \in R^{n+}} \left\| \sum_{i=1}^{n} (c_i + \varepsilon\, d_i)\, x_i \right\| \tag{37.1}$$

under strong constraints:

$$\sum_{i=1}^{n} (a_{ki} + \varepsilon\, \alpha_{ki})\, x_i \,\breve{\ge}\, b_k + \varepsilon \beta_k \quad k \in \{1, \cdots, m\} \tag{37.2}$$

$$\text{and} \quad x_i \in R^+ \quad i \in \{1, \cdots, n\} \tag{37.3}$$

In this case uncertainty is attached to cost coefficients $c_i$, to technical parameters $a_{ki}$ and to constraint levels $b_k$.

The above problem corresponds to the minimization of the worst estimate of total cost with satisfaction of strong level constraints. Here variables $x_i$ are supposed to be real positive

but they could be either fully real or integer. In the case in which the $d_i$ are zero, the fuzziness is restricted to the feasible set.
Problem $D_0$ is equivalent to the following problem in $R^{+n}$ :

$$\min_{\underline{x}\in R^{n+}} \left|\sum_{i=1}^{n} c_i x_i\right| + \rho \sum_{i=1}^{n} d_i x_i \tag{38.1}$$

under the constraints:

$$\sum_{i=1}^{n} (a_{ki} - \rho\,\alpha_{ki}) x_i \geq b_k + \rho\,\beta_k \quad k \in \{1,\cdots,m\} \tag{38.2}$$

$$\text{and} \quad x_i \geq 0 \quad i \in \{1,\cdots,n\} \tag{38.3}$$

Then it can be seen that the proposed formulation leads to minimize a combination of the values of the nominal criterion and of its degree of uncertainty. In the case in which the cost coefficients are positive this problem reduces to a classical linear programming problem over $R^{+n}$. In the general case, since the quantity $\sum_{i=1}^{n} c_i x_i$ will have at solution a particular sign, the solution $\underline{x}*$ of problem $D_0$ will be such as:

$$\arg\min\{\ \min_{\underline{x}\in R^{n+}}(\sum_{i=1}^{n} c_i \bar{x}_i + \rho \sum_{i=1}^{n} d_i \bar{x}_i), \min(\rho \sum_{i=1}^{n} d_i \vec{x}_i - \sum_{i=1}^{n} c_i \vec{x}_i)\ \} \tag{39}$$

where $\underline{\bar{x}}$ is solution of problem:

$$\min_{\underline{x}\in R^{n+}} (\sum_{i=1}^{n} c_i x_i + \rho \sum_{i=1}^{n} d_i x_i) \tag{40.1}$$

under the constraints:

$$\sum_{i=1}^{n} (a_{ki} - \rho\ \alpha_{ki}) x_i \geq b_k + \rho\,\beta_k \quad k \in \{1,\cdots,m\} \tag{40.2}$$

$$\sum_{i=1}^{n} c_i x_i \geq 0 \quad \text{and} \quad x_i \geq 0 \quad i \in \{1,\cdots,n\} \tag{40.3}$$

and where $\underline{\vec{x}}$ is solution of problem:

$$\min_{\underline{x}\in R^{n+}} (\rho \sum_{i=1}^{n} d_i x_i - \sum_{i=1}^{n} c_i x_i) \tag{41.1}$$

under the constraints:

$$\sum_{i=1}^{n} (a_{ki} - \rho\,\alpha_{ki}) x_i \geq b_k + \rho\,\beta_k \quad k \in \{1,\cdots,m\} \tag{41.2}$$

$$\sum_{i=1}^{n} c_i x_i \leq 0 \quad \text{and} \quad x_i \geq 0 \quad i \in \{1,\cdots,n\} \tag{41.3}$$

The fuzzy dual optimal performance of this program will be given by:

$$\sum_{i=1}^{n} (c_i + \varepsilon\, d_i) x_i^* = \sum_{i=1}^{n} c_i x_i^* + \varepsilon \sum_{i=1}^{n} d_i x_i^* \tag{42}$$

Problems (40) and (41) are here again classical linear programming problems. Considering other linear constraints involving the other partial order relations over $\tilde{\Delta}$ (weak inequality and fuzzy equality) the solution of the fuzzy dual programming problem will lead to the consideration of at most two classical linear programming problems. The integer version of problem $D_0$ will lead also to classical integer linear programming problems.

## 6. Fuzzy dual programming with fuzzy dual parameters and variables

Now we consider fuzzy dual programming problems with fuzzy dual variables. In that case we formulate problem $D_1$ :

$$\min_{\underline{x}\in R^n,\underline{y}\in R^{n+}} \left\| \sum_{i=1}^{n} (c_i + \varepsilon\, d_i)(x_i + \varepsilon\, y_i) \right\| \tag{43.1}$$

under the strong constraints :

$$\sum_{i=1}^{n} (a_{ki} + \varepsilon\, \alpha_{ki})(x_i + \varepsilon\, y_i) \breve{\geq} b_k + \varepsilon \beta_k \quad k \in \{1,\cdots,m\} \tag{43.2}$$

$$\text{and} \quad x_i \in R,\ y_i \geq 0 \quad i \in \{1,\cdots,n\} \tag{43.3}$$

The above problem corresponds to the minimization of the worst estimate of total cost with satisfaction of strong level constraints when there is some uncertainty not only on the values of the parameters but also on the capability to implement exactly the best solution. Problem $D_1$ can be rewritten as :

$$\min_{\underline{x}\in R^n,\underline{y}\in R^{n+}} \left\| \sum_{i=1}^{n} (c_i\ x_i + \varepsilon(|x_i| d_i + |c_i| y_i) \right\| \tag{44.1}$$

under constraints 43.3 and :

$$\sum_{i=1}^{n} (a_{ki}\ x_i + \varepsilon\, (\alpha_{ki} |x_i| + |a_{ki}| y_i)) \breve{\geq} b_k + \varepsilon \beta_k \quad k \in \{1,\cdots,m\} \tag{44.2}$$

which is equivalent in $R^n \times R^{n+}$ to the following mathematical programming problem:

$$\min_{\underline{x}\in R,\underline{y}\in R^{n+}} C(\underline{x},\underline{y}) = \left| \sum_{i=1}^{n} c_i x_i \right| + \rho \sum_{i=1}^{n} (d_i |x_i| + |c_i| y_i) \tag{45.1}$$

under constraints 43.3 and:

$$\sum_{i=1}^{n} (a_{ki}\ x_i - \rho(\alpha_{ki} |x_i| + |a_{ki}| y_i)) \geq b_k + \rho\, \beta_k \quad k \in \{1,\cdots,m\} \tag{45.2}$$

Let

$$A(\underline{x},\underline{y}) = \left\{ \underline{x} \in R^n, \underline{y} \in R^{+n} : \sum_{i=1}^{n} (a_{ki}\ x_i - \rho(\alpha_{ki} |x_i| + |a_{ki}| y_i)) \geq b_k + \rho\, \beta_k \quad k \in \{1,\cdots,m\} \right\} \tag{46}$$

since

$$\forall x \in R^n, \forall y \in R^{n+} \qquad A(\underline{x},\underline{y}) \subset A(\underline{x},\underline{0}) \ \text{ and } \ C(\underline{x},\underline{y}) \geq C(\underline{x},\underline{0}) \tag{47}$$

it appears, as expected, that the case of no diversion of the nominal solution is always preferable. In the case in which the diversion from the nominal solution is fixed to $\bar{y}_i, i \in \{1,\cdots,n\}$, problem $D_1$ has the same solution than problem $D_1{}'$ :

$$\min_{\underline{x}\in R^n} \left| \sum_{i=1}^{n} c_i x_i \right| + \rho \sum_{i=1}^{n} d_i |x_i| \tag{48.1}$$

under constraints 43.3 and :

$$\sum_{i=1}^{n}(a_{ki}\,x_i-\rho\,\alpha_{ki}|x_i|)\geq b_k+\rho(\beta_k+\sum_{i=1}^{n}|a_{ki}|\,\bar{y}_i)\quad k\in\{1,\cdots,m\}\qquad(48.2)$$

The fuzzy dual optimal performance of problem (43) will be given by:

$$\sum_{i=1}^{n}c_i\;x_i^*+\varepsilon\sum_{i=1}^{n}\;(|x_i^*|\,d_i+|c_i|\,y_i)\qquad(49)$$

where $\underline{x}*$ of problem $D_0^{'}$.

In the case in which *p* of the *n* decision variables are of undetermined sign, the solution of this problem can be obtained by solving $2^{p+1}$ classical linear programming problems. Here also other linear constraints involving the other partial order relations over $\tilde{\Delta}$ (weak inequality and fuzzy equality) could be introduced in the formulation of problem $D_1$ while the consideration of he integer version of problem $D_1$ will lead also to solve families of classical integer linear programming problems.

The performance of the solution of problem $D_1$ will be potentially diminished by the reduction of the feasible set defined by (42.3) and (47.2).

## 7. Conclusion

In this study to treat general programming problems presenting some uncertainty on the values of parameters and variables, we introduced a special class of fuzzy numbers, fuzzy dual numbers. Fuzzy dual calculus was developed in such a way that the interpretation of their dual part as an uncertainty level remains valid through the basic operations on these numbers.

Then it has been shown how the induced pseudo norm allows to set partial orders of different intensity between fuzzy dual numbers. This enables a rather simple definition of fuzzy feasible sets through fuzzy dual constraints. To display the generality of the proposed approach, the case of fuzzy dual LMI's has also been considered since many mathematical programming problems adopt the LMI representation for their feasible sets.

Then, fuzzy dual programming problems with either uncertain parameters or variables have been considered. Although only the linear case has been considered in this study, it appears that the proposed approach to treat uncertainty leads for the solution of a fuzzy dual programming problem to the consideration of a finite collection of classical mathematical programming problems.

Finally, it can be considered that the proposed approach provides a way to tackle a large class of stochastic optimization problems with an new trade-off between the accuracy of the representation of uncertainty and the resulting computer burden.

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