TWO STAGE FACILITY LOCATION PROBLEM: LAGRANGIAN BASED HEURISTICS

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ABSTRACT

In the two-stage capacitated facility location problem a single product is produced at some plants in order to satisfy customer demands. The product is transported from these plants to some depots and then to the customers. The capacities of the plants and depots are limited. The aim is to select cost minimizing locations from a set of potential plants and depots. This cost includes fixed cost associated with opening plants and depots, and variable cost associated with both transportation stages. In this work, several Lagrangian relaxations are analyzed and compared, a Lagrangian heuristic producing feasible solutions is presented. The results of a computational study are reported.

KEYWORDS. Two stage facility location problem, Lagrangian Relaxation, Lagrangian Heuristic, Subgradient Technique.

Main area: Mathematical Programming.

1. Introduction

The two-stage capacitated facility location problem can be defined as follows: a single product is produced at plants and then transported to depots, both having limited capacities. From the depots the product is transported to customers to satisfy their demands. The use of the plants/depots incurs a fixed cost, while transportation from the plants to the customers through the depots results in a variable cost. We need to identify what plants and depots to use, as well as the product flows from the plants to the depots and then to the customers such that the demands are met at a minimal cost.

Facility location problems have numerous applications and have been widely studied in the literature, see the review publications by Daskin, Snyder & Berger (2003), Klose & Drexl (2004) and Melo, Mickel & Saldanha-da-Gama (2009) and the references therein. Various applications of the facility location in supply chain optimization and management are presented in Wang (2011) and Minis et al. (2011).

Various exact approaches have been proposed for the location problems. For example, Avella & Boccia (2007) presented a family of minimum knapsack inequalities of a mixed type, containing both binary and continuous (flow) variables for the capacitated problem and developed a branch and cut and price algorithm to deal with large scale instances. Klose & Drexl (2005) considered a new lower bound for the capacitated facility location problem based on partitioning the plant set and employing column generation.

Approximate approaches can be roughly divided into two large groups: metaheuristics and Lagrangian based techniques. Metaheuristic approaches to the problem like tabu search, GRASP, are discussed in Filho & Galvão (1998). An algorithm for large instances is presented in Barahona & Chudack (2005), they used a heuristic procedure that produces a feasible integer solution and used a Lagrangian relaxation to obtain a lower bound on the optimal value.

A Lagrangian based heuristic for solving the capacitated plant location problem with side constraints was presented in Sridharan (1991). Approaches and relaxations proposed in the literature for the capacitated facility location problem are compared in Cornuejols, Sridharan & Thizy (1991). A linear programming based heuristic is considered in Klose (1999) for a two-stage capacitated problem with single source constraints. Wollenweber (2008) proposed a greedy construction heuristic and a Variable Neighborhood Descent and a Variable Neighborhood Search for the multi-stage facility location problem with staircase costs and splitting of commodities. In Landete & Marín (2009) the asymmetry inherent to the problem in plants and depots is taking into account to strengthening the formulation. Gendron & Semet (2009) presented two formulations for the problem and compared the linear relaxation of each formulation and the binary relaxation of the model.

Several Lagrangian relaxation approaches have been proposed for the two stage facility location problem. For the uncapacitated case Chardaire, Lutton & Sutter (1999) studied the effectiveness of the formulation for the two level simple plant location problem incorporating polyhedral cuts and proposed an approach combining a Lagrangian relaxation method and a simulated annealing algorithm. Lu & Bostel (2005) proposed an algorithm based on Lagrangian heuristics for a 0-1 mixed integer model of a two level location problem with three types of facility to be located. In Marín (2007) a mixed integer formulation and several Lagrangian relaxations to determine lower bounds for the two stage uncapacitated facility location problem are presented.

The Lagrangian relaxation for the capacitated case was studied and numerically tested in Barros & Labbé (1994). Bloemhof et al. (1996) studied alternative model formulations of the capacitated problem obtaining lower bounds by Lagrangian relaxations of the flow-balancing constraints. They also developed heuristic procedures to obtain feasible solutions. In Marín & Pelegrín (1999) several Lagrangian relaxations for two different formulations of the two-stage problem are computationally compared. Tragantalerngsak et al. (1999) proposed a Lagrangian relaxation-based branch and bound algorithm for the two-echelon, single source, capacitated problem. A Lagrangian heuristic is proposed in Klose (2000) using relaxation of the capacity constraints for the problem with a fixed number of plants. Feasible solutions are constructed from those of the Lagrangian sub problems by applying simple reassignment procedures.

In many techniques Lagrangian relaxation is used in twofold: the optimal value of the Lagrangian (dual) problem is used as a dual bound, while the Lagrangian solution is used as a starting or reference point to produce a feasible solution and a corresponding primal bound. Frequently a relaxation is considered as *good* if it produces a tight dual bound. Meanwhile, the quality of the feasible Lagrangian based solution has also to be taken into account in evaluating the relaxation. There are often different ways in which a given problem can be relaxed in a Lagrangian fashion. It is unlikely to highlight a single relaxation producing high quality bounds of both types, primal and dual. Moreover, if the quality of the dual bound is basically defined by the constraints relaxed, the quality of the primal bound depends also on the algorithm used to restore the feasibility of the Lagrangian solution.

In this paper we consider a simple decomposable relaxation and an algorithm to restore the feasibility of the corresponding Lagrangian solution. This relaxation produces a poor dual bound and never been considered before as a promising relaxation. Meanwhile, the relaxation results in a very tight feasible solution typically within 0.5-1.0% of the relative sub optimality.
The rest of the paper is organized as follows. The next section presents a mathematical formulation for the two-stage facility location problem and the Lagrangian bound. A heuristic procedure to get feasible solutions is presented in section 3. Computational results are reported in section 4, while section 5 concludes.

2. Problem formulation and Lagrangian bound

To formally describe the problem, let $I = 1,\dots,n$ be the index set of potential plants, $J = 1,\dots,m$ the index set of potential depots and $K=1,\dots,k$ the index set of clients. Then, the problem can be formulated as the following mixed integer linear program:

$$w = mín \sum_{i\in I} f_i y_i + \sum_{j\in J} g_j z_j + \sum_{(i\in I, j\in J)} c_{ij} x_{ij} + \sum_{(j\in J, k\in K)} d_{jk} s_{jk} \qquad (1)$$

$$s.t.: \quad \sum_{j\in J} x_{ij} \le b_i \qquad ; i \in I, \qquad (2)$$

$$\sum_{i\in I} x_{ij} \le p_j \qquad ; j \in J, \qquad (3)$$

$$\sum_{j\in J} s_{jk} \ge q_k \qquad ; k \in K, \qquad (4)$$

$$\sum_{i\in I} x_{ij} \ge \sum_{k\in K} s_{jk} \qquad ; j \in J, \qquad (5)$$

$$x_{ij} \le m_{ij} y_i \qquad ; i \in I, j \in J, \qquad (6)$$

$$s_{jk} \le l_{jk} z_j \qquad ; j \in J, k \in K, \qquad (7)$$

$$x_{ij}, s_{jk} \in R^+, y_i, z_j \in \{0,1\} \qquad (8)$$

Here f_i and g_j are the fixed costs associated with the installation of plant *i* and depot *j*; c_{ij} and d_{jk} are the costs of transportation from plant *i* to depot *j* and from depot *j* to client *k*, respectively; q_k is the demand of client *k*; while b_i and p_j are the capacities of the corresponding plant and depot. The variables in this formulation are $y_i = 1$ if plant *i* is installed and $y_i = 0$ otherwise, $z_j = 1$ if depot *j* is installed and $z_j = 0$ otherwise, x_{ij}, s_{jk} are the transportation flows between the corresponding units.

Constraints (2) and (3) represent capacity limits for plants and depots, (4) is the demand constraint (for each customer, at least the demand must be met), (5) is the relaxed flow conservation constraint (the product transported from the depot must at least be transported to it from the plants), constraints (6) and (7), together with (8), assure that there is a flow only from plants and depots installed. Constants m_{ij}, l_{jk} represent the upper bounds for the respective flows, we may set, e.g., $m_{ij} = min\{b_i, p_j\}$; $l_{jk} = min\{p_j, q_k\}$. Note that by minimizing the objective (1), constraints (5) are fulfilled as equalities for an optimal solution of (1)-(8).

Constraints (6), (7) can be stated in a more compact form yielding an equivalent formulation of the two stage location problem (formulation B):

$$w = mín\sum_{i\in I} f_i y_i + \sum_{j\in J} g_j z_j + \sum_{(i\in I, j\in J)} c_{ij}x_{ij} + \sum_{(j\in J, k\in K)} d_{jk}s_{jk} \tag{9}$$

$$s.t.: \quad \sum_{j\in J} s_{jk} \geq q_k \qquad ; k\in K, \tag{10}$$

$$\sum_{i\in I} x_{ij} \geq \sum_{k\in K} s_{jk} \qquad ; j\in J, \tag{11}$$

$$\sum_{j\in J} x_{ij} \leq b_i y_i \qquad ; i\in I, \tag{12}$$

$$\sum_{i\in I} x_{ij} \leq p_j z_j \qquad ; j\in J, \tag{13}$$

$$x_{ij}, s_{jk} \in R^+, y_i, z_j \in \{0,1\} \tag{14}$$

Constraints (12) together with (14) assure the outflows only from plants opened, while constraints (13) together with (11) assure the flows only from and to depots opened.

Formulations A and B are equivalent in the sense that both result in the same optimal solution. However, they have different polyhedral structure of the feasible sets and thus we may expect that relaxing the same constraints may result in different values for the corresponding Lagrangian bounds.

Lagrangian bounds are widely used as a core of many numerical techniques, e.g. in branch and bound schemes for integer and combinatorial problems. Most Lagrangian relaxation approaches for the capacitated facility location problem are based either on dualizing the demand constraints or the depot capacity constraints (Klose, 1999).

For the formulation A defined by (1)-(8) five "decomposable" Lagrangian relaxations are considered, denoted as follows (all Lagrangian multipliers are assumed to be nonnegative):

RA1: constraints (5), representing interconnections between the two stages, are dualized giving a Lagrangian problem that can be decomposed into a sub problem for the first stage (plants) in variables x, y and a sub problem corresponding to the second stage (depots) in s, z .

RA2: constraints (6) are dualized giving the Lagrangian problem that can be decomposed into $|I|$ subproblems of the form $w_{1i}^{RA2} = mín\{y_i(f_i - u_{ij}m_{ij}),\ y_i \in \{0,1\}\}$, which can be analyzed analytically and a sub problem in variables x, s, z .

RA3: constraints (7) are dualized. Similar to RA2 we get $|J|$ sub problems in variables z and a sub problem in *(x,s,y)*.

RA4: constraints (2) and (4), binding with respect to index *j*, are dualized. The Lagrangian problem then decomposes into $|J|$ independent sub problems.

RA5: constraints (3) and (5) are dualized. Similar to RA1 this Lagrangian problem decomposes into a sub problem corresponding to the first stage (plants), and a sub problem corresponding to the second stage (depots).

The Lagrangian relaxations for the formulation B defined by (9)-(14) are as follows:

RB1: constraints (11) are dualized giving two Lagrangian sub problems: a sub problem in *(x,y,z)*, and a sub problem in *s*. The latter problem is decomposed into $|K|$ independent continuous one-dimensional knapsack problems which can be analyzed analytically.

RB2: constraints (12) are dualized, resulting in $|I|$ independent sub problems of the form $w_{1i}^{RB2} = mín\{y_i(f_i - u_i b_i),\ y_i \in \{0,1\}\}$ with only one binary variable and a sub problem in (*x; s; z*).

RB3: constraints (13) are relaxed giving similar to RB2 $|J|$ independent sub problems in *z* and a sub problem in (*x; s; y*).

RB4: constraints (10) and (12) are dualized. The Lagrangian problem then decomposes into $|I|$ independent sub problems of the form $w_{1i}^{RB4} = mín\{y_i(f_i - v_i b_i),\ y_i \in \{0,1\}\}$, that can be analyzed analytically, and a sub problem in (*x; s; z*). The latter decomposes into $|J|$ independent sub problems with only one binary variable and can be solved by inspection (see, e.g., Wolsey, 1999), fixing z to 0 or 1 and then solving the remaining problem with continuous variables *(x,s)*.
RB5: constraints (11) and (13) are dualized, the Lagrangian problem decomposes into three types of sub problems. We have $|J|$ independent sub problems in z of the form $w_j^{RB5} = mín\{z_j(g_j - v_j p_j),\ z_j \in \{0,1\}\}$. We also have $|K|$ independent continuous one-dimensional knapsack sub problems in s, and $|I|$ independent sub problems in x, y. The latter problem has only one binary variable and can be solved by inspection.

The problem of finding the best, *i.e.* bound maximizing Lagrange multipliers, is called the Lagrangian dual. To solve the Lagrangian dual problem one can apply a constraint generation scheme (Benders method) transforming the dual problem into a large-scale linear programming problem. The main advantage of using Benders technique is that it generates two-sided estimations for the dual bound in each iteration thus producing near-optimal dual bound with guaranteed quality. Meanwhile, the computational cost of this scheme is typically high. Another popular approach to solve the dual problem is by subgradient optimization. In contrast to the Benders method, the subgradient technique does not provide the value of the bounds with the prescribed accuracy. That is, terminating iterations of the subgradient method using some stopping criteria we can expect only approximate values of the bound. We do not consider here these two well-known approaches in details, referring the reader to Lasdon(1970) , Wolsey (1999) and Conejo (2000) for the constraint generation (Benders) technique, and to Martin (1999) and Guignard (2003) for the subgradient scheme.

4. Getting feasible solutions

To get a feasible solution from the Lagrangian one we use a simple algorithm to recover feasibility. In fact, this approach can be applied to any nonfeasible solution.
Let $\bar{x}_{ij}, \bar{s}_{jk}$ be a nonfeasible solution

Do $y_i = 0, \forall i, I_1 = \varnothing, I_0 = I$;

$z_j = 0, \forall j, J_1 = \varnothing, J_0 = J$.

Step 0: Do $y_i \leftarrow \dfrac{\sum_{j\in J} \bar{x}_{ij}}{b_i}, z_j \leftarrow \dfrac{\sum_{k\in K} \bar{s}_{jk}}{p_j}$.

Step 1: $i^* = \arg max\{\, y_i | i \in I_0\}$.

Step 2: $y_{i^*} \leftarrow 1, I_1 \leftarrow I_1 \cup \{i^*\}, I_0 \leftarrow I_0 - \{i^*\}$.

Step 3: If $\sum_{i\in I_1} b_i \geq \sum_{k\in K} q_k$ go to step 4 and do $y_i = 0, \forall i \in I_0$, otherwise, return to step 1.

Step 4: $j^* = \arg max\{\, z_j | j \in J_0\}$.

Step 5: $z_{j^*} \leftarrow 1, J_1 \leftarrow J_1 \cup \{j^*\}, J_0 \leftarrow J_0 - \{j^*\}$.

Step 6: If $\sum_{j\in J_1} p_j \geq \sum_{k\in K} q_k$ go to step 7 and do $z_j = 0, \forall j \in J_0$, otherwise, return to step 4.

Step 7: Fix y_i and z_j in the original problem and solve the corresponding linear problem to obtain the flows.

In this algorithm we calculate for each plant a "saturation" indicator representing the relative usage of its capacity (step 0). Then the plant having the highest saturation is opened (step 1). If the capacity is sufficient to satisfy the total customers' demand, the rest of the plants are closed, otherwise the plant having the next highest indicator is opened too (steps 2 and 3). The depots are opened in a similar way (steps 3, 5 and 6). Fixing the binary variables obtained by this procedure, the flows are determined from the corresponding linear problem.

5. Computational results

A numerical study for the two-stage capacitated facility location problem was conducted to compare the bounds. The following sets of instances were generated according to the values (I; J; K): A(3; 5; 9); B(5; 7; 30); C(7; 10; 50); D(10; 10; 100); E(10;16;30); F(30;30;30); G(30;36;120); H(30;30;100); I (50;50;200). Every set contains 20 problem instances. The data were random integers generated as follows:

$$c_{ij}, d_{jk} \in U[10,20],\ \mathrm{q}_k \in U[1,10],$$

$$b_i \in \left\lceil 10\frac{J+K}{I} \right\rceil + U[0,10],\ p_j \in \left\lceil 10\frac{K}{J} \right\rceil + U[0,10]$$

Two different ways to generate the fixed costs were implemented. For the first ten instances in each class the fixed costs f_i,g_j were random integers generated independently on the number clients, plants and depots: $f_i, g_j \in U[100,200]$. For the remaining ten instances the fixed costs *fi* for plants were proportional to the number of depots and clients, while the fixed costs *gj* for depots were proportional to the number of clients:

$$f_i \in \left\lceil 100\frac{K+J}{I} \right\rceil + U[0,100];\ g_j \in \left\lceil 100\frac{K}{J} \right\rceil + U[0,100]$$

The dual bound corresponding to the Lagrangian relaxation was calculated by the subgradient technique. In each iteration of this method the feasible solution was obtained by the Algorithm.

The best (over all iterations) feasible solution was stored. The current best feasible solution was used to update the step size. If after 5 consecutive iterations of the subgradient technique the dual bound was not improved, the half of the step size scaling parameter was used. The process stops if the step size scaling parameter is less than 0:0001, or if the maximum number (300) of iterations is reached. The procedure was implemented in GAMS/CPLEX 11.2 using a Sun Fire V440 terminal, connected to 4 processors Ultra SPARC III with 1602 Hhz, 1 MB of CACHE, and 8 GB of memory.

For all the instances we have calculated:

- z_{IP} – the value of the optimal objective of the two stage location problem.
- z_L - the value of the best Lagrangian bound.
- z_{BF} - the objective value corresponding to the best feasible solution.

The relative quality of the Lagrangian bound and of the best feasible solution was measured by

$$\varepsilon_L = \frac{z_{IP} - z_L}{z_{IP}} \times 100\% \text{ and } \varepsilon_{BF} = \frac{z_{BF} - z_{IP}}{z_{BF}} \times 100\%,$$

correspondingly. The similar proximity indicators are used to measure the quality of the bounds and feasible solutions derived from other relaxations and other feasible solutions.
The results obtained for the Type 1 instances are presented in Table 1.

Table 1 – Type 1 instances.

	Lagrangian relaxations						Feasible solutions					
	In the top 3 (%)			Best bound (%)			In the top 3 (%)			Best bound (%)		
Size	RB1	RB2	RB3	RB1	RB2	RB3	RA2	RA3	RB4	RA2	RA3	RB4
A	70	80	40	10	0	20	70	90	90	70	90	80
B	70	80	70	10	20	60	20	90	80	10	80	50
C	90	80	50	20	40	30	20	80	90	10	60	40
D	70	80	10	40	10	0	20	90	90	20	40	50

The first group of columns in Table 1 represents (in %) how many times the corresponding dual bound appeared among the best 3 bounds. The second group of columns shows (in %) how many times the corresponding dual bound was the best. The indicators are presented only for the dual bounds corresponding to RB1, RB2 and RB3 since they were most frequently among the best. The last columns present indicators for the best feasible solutions obtained by the Algorithm in the course of solving the dual problem. The third group of columns in Table 1 represents (in %) how many times the corresponding feasible solution appeared among the best 3 solutions. The last group of columns shows (in %) how many times the corresponding feasible solution was the best. The indicators are presented only for the feasible solutions derived from RA2, RA3, and RB4 since they were most frequently among the best. Table 2 presents similar results for the Type 2 instances.

Table 2 – Type 2 instances.

	Lagrangian relaxations						Feasible solutions					
	In the top 3 (%)			Best bound (%)			In the top 3 (%)			Best bound (%)		
Size	RB1	RB2	RB3	RB1	RB2	RB3	RA2	RA3	RB4	RA2	RA3	RB4
A	80	90	100	0	10	90	90	90	60	90	90	60
B	100	100	100	10	10	80	90	50	50	60	40	20
C	100	100	100	20	20	60	80	70	70	40	30	40
D	100	100	100	40	20	40	90	40	90	70	10	40

As can be seen from Tables 1 and 2, the bound corresponding to RB3 appears frequently among the best Lagrangian bounds. Considering the quality of the feasible solutions we may highlight RB4 which is frequently among the best and is easier to calculate than RA3. We note that relaxations giving the best dual bounds were never among those producing the best feasible solutions.

In the Lagrangian problem corresponding to RB4 the demand and the capacity plant constraints are relaxed. The problem decomposes into the following sub problems:

RB4: constraints (10) and (12) are dualized:

$$w^{RB4} = mín \sum_{i\in I} f_i y_i + \sum_{j\in J} g_j z_j + \sum_{(i\in I, j\in J)} c_{ij} x_{ij} + \sum_{(j\in J, k\in K)} d_{jk} s_{jk} + \sum_{k\in K} u_k (q_k - \sum_{j\in J} s_{jk}) + \sum_{i\in I} v_i (\sum_{j\in J} x_{ij} - b_i y_i)$$

$$\sum_{i\in I} x_{ij} \geq \sum_{k\in K} s_{jk} \qquad ; j \in J,$$

$$\sum_{i\in I} x_{ij} \leq p_j z_j \qquad ; j \in J,$$

$$x_{ij}, s_{jk} \in R^+, y_i, z_j \in \{0,1\}$$

The Lagrangian problem then decomposes into $|I|$ independent sub problems of the form $w_{1i}^{RB4} = mín\{y_i(f_i - v_i b_i), y_i \in \{0,1\}\}$, that can be analyzed analytically, and a sub problem in (*x; s; z*):

$$w_2^{RB4} = mín \sum_{j \in J} g_j z_j + \sum_{(i \in I, j \in J)} (c_{ij} + v_i) x_{ij} + \sum_{(j \in J, k \in K)} (d_{jk} - u_k) s_{jk}$$
$$\sum_{i \in I} x_{ij} \geq \sum_{k \in K} s_{jk} \qquad ; j \in J,$$
$$\sum_{i \in I} x_{ij} \leq p_j z_j \qquad ; j \in J,$$
$$x_{ij}, s_{jk} \in R^+, z_j \in \{0,1\} \quad .$$

The latter decomposes into $|J|$ independent sub problems of the form:

$$w_j^{RB4} = mín\ g_j z_j + \sum_{i \in I} (c_{ij} + v_i) x_{ij} + \sum_{k \in K} (d_{jk} - u_k) s_{jk}$$
$$\sum_{i \in I} x_{ij} \geq \sum_{k \in K} s_{jk}$$
$$\sum_{i \in I} x_{ij} \leq p_j z_j$$
$$x_{ij}, s_{jk} \in R^+, z_j \in \{0,1\} \quad .$$

This problem has only one binary variable and can be solved by inspection (see, e.g., Wolsey, 1999), fixing z to 0 or 1 and then solving the remaining problem with continuous variables *(x,s)*.

Thus we may conclude that the computational cost to solve the Lagrangian problem corresponding to the relaxation RB4 is very low, in fact no integer problem is involved.

Table 3 present the results obtained for the first way to generate data for the RB4 relaxation, while Table 4 gives the results for the second way to generate data for the same relaxation. The results are shown for 5 different instances for each problem size. The first two columns present the proximity indicators for the corresponding dual bound and for the best (over all iterations) feasible solution obtained by the Algorithm. The last two columns give the proximity indicator for the feasible solution corresponding to the last and the first iteration of the subgradient technique. The number in the parenthesis indicates the number of the iteration corresponding to the bound value.

Table 3 – Results for RB4 type 1.

Size	ε_L (%)	ε_{BF} (%)	ε_{LF} (%)	ε_{FF} (%)
A1	12.30	1.81 (14)	5.07 (135)	12.51
A2	2.31	0.00 (31)	0.00 (89)	14.95
A3	7.74	0.57 (5)	4.21 (138)	4.21
A4	6.39	0.00 (9)	0.00 (128)	11.63
A5	11.03	0.00 (4)	3.79 (130)	3.79
B1	6.04	0.00 (27)	0.99 (124)	3.47
B2	4.03	0.00 (17)	0.00 (135)	2.35
B3	5.93	1.26 (16)	4.73 (132)	2.66
B4	4.97	0.00 (15)	4.91 (147)	4.18
B5	4.15	0.18 (28)	0.18 (102)	8.37
C1	4.89	0.00 (36)	1.78 (118)	3.39
C2	4.43	0.35 (9)	2.81 (116)	8.22
C3	2.88	0.03 (8)	2.28 (151)	7.59
C4	4.59	0.00 (28)	6.57 (116)	8.49
C5	3.03	0.00 (28)	2.43 (118)	6.91
D1	2.89	0.00 (16)	4.26 (113)	5.30
D2	2.43	0.00 (24)	3.14 (129)	7.44
D3	3.75	0.00 (96)	5.55 (131)	9.71
D4	3.70	0.49 (1)	2.16 (130)	4.19

D5	2.07	0.00 (105)	1.75 (153)	3.79

Table 4 – Results for RB4 type 2.

Size	ε_L (%)	ε_{BF} (%)	ε_{LF} (%)	ε_{FF} (%)
A1	13.64	0.51 (9)	4.72 (214)	10.31
A2	4.79	0.00 (29)	0.00 (166)	14.48
A3	5.33	0.43 (2)	13.46 (158)	18.41
A4	6.89	0.00 (28)	10.17 (188)	11.98
A5	19.09	0.00 (1)	0.00 (157)	0.00
B1	13.18	0.00 (137)	0.79 (212)	2.09
B2	9.09	0.00 (58)	1.62 (250)	1.44
B3	8.29	1.57 (22)	9.79 (250)	8.63
B4	5.49	0.00 (61)	6.01 (243)	6.01
B5	9.32	0.11 (21)	0.11 (252)	14.79
C1	6.68	0.00 (62)	8.28 (223)	9.95
C2	10.83	0.25 (181)	2.11 (270)	5.06
C3	6.84	0.56 (76)	2.28 (258)	4.64
C4	8.55	0.00 (27)	4.80 (237)	5.28
C5	6.47	0.00 (47)	1.49 (250)	4.34
D1	6.32	0.20 (62)	2.12 (284)	6.79
D2	6.08	0.40 (237)	4.34 (256)	4.48
D3	8.48	0.00 (67)	1.62 (277)	5.64
D4	7.29	1.91 (18)	8.18 (250)	8.45
D5	5.57	0.19 (153)	3.58 (276)	2.19

Along with the set instances A – D we have used larger instances generated according to the values (*I; J; K*):

- E(10; 16; 30);
- F(30; 30; 30);
- G(30; 60; 120);
- H(30; 30; 100);
- I(50; 50; 200).

Table 5 – Results for RB4 type 1.

Size	ε_L (%)	ε_{BF} (%)	ε_{LF} (%)	ε_{FF} (%)
E1	4.56	0.00 (59)	0.10 (225)	11.65
E2	5.03	0.53 (92)	0.75 (217)	7.79
E3	3.94	1.37 (49)	5.67 (231)	12.86
E4	3.47	0.52 (134)	2.05 (256)	9.14
E5	4.25	0.46 (191)	6.07 (256)	9.10
F1	2.21	0.75 (109)	1.44	13.92
F2	1.89	0.00 (53)	0.89 (235)	13.67
F3	1.88	0.03 (58)	1.17 (198)	15.48
F4	1.61	1.55 (264)	3.67	19.47
F5	1.91	0.55 (60)	1.66 (266)	13.71
G1	1.10	0.37 (240)	1.91	8.11
G2	0.99	1.18 (88)	1.58	7.63
G3	1.13	0.21 (249)	1.24	8.69
G4	0.94	0.56 (100)	1.81	7.42

G5	0.67	0.23 (194)	0.87	9.92
H1	1.04	0.46 (284)	2.02	10.28
H2	1.29	0.47 (168)	1.04	5.84
H3	1.49	0.38 (191)	1.57	8.18
H4	1.39	0.77 (284)	2.57	7.16
H5	1.73	0.73 (147)	3.93	8.83
I1	0.96	0.52 (199)	1.27	5.98
I2	0.93	0.81 (130)	2.52	6.44
I3	0.78	1.09 (196)	1.49	7.01
I4	0.73	0.92 (202)	1.57	5.94
I5	0.64	1.04 (94)	1.90	6.44

Table 6 – Results for RB4 type 2.

Size	ε_L (%)	ε_{BF} (%)	ε_{LF} (%)	ε_{FF} (%)
E1	4.04	2.54 (44)	9.06 (227)	13.75
E2	9.56	0.79 (71)	0.79 (205)	4.93
E3	3.71	5.76 (51)	6.83 (293)	14.73
E4	6.25	0.00 (108)	1.69 (233)	7.35
E5	7.40	0.32 (56)	8.03 (277)	8.79
F1	3.15	0.67 (133)	1.31 (257)	13.41
F2	2.94	0.00 (60)	1.14 (261)	15.65
F3	2.28	0.95 (56)	2.93 (250)	14.24
F4	2.90	0.93 (51)	2.06 (261)	13.58
F5	2.96	0.64 (32)	5.42 (277)	20.07
G1	3.33	0.61 (194)	1.12	6.94
G2	1.98	0.82 (128)	1.56	8.22
G3	1.27	0.46 (120)	2.60	8.29
G4	0.87	0.16 (260)	1.11	7.15
G5	1.51	0.70 (87)	1.87	9.14
H1	1.64	1.16 (206)	2.52	8.33
H2	2.23	0.58 (214)	2.88	8.28
H3	1.54	0.23 (240)	2.25	7.26
H4	1.47	0.58 (127)	3.49	5.37
H5	1.14	0.43 (78)	2.64	8.81
I1	1.09	0.79 (211)	1.09	6.61
I2	1.95	0.81 (207)	2.27	5.62
I3	1.59	1.41 (221)	1.81	5.28
I4	1.29	1.01 (168)	2.89	6.59
I5	1.48	0.94 (146)	1.78	5.67

As we can see from the Tables for both ways to generate the data the approach provides very tights feasible solutions, typically within 0.5-1.0% of the relative proximity. So we may expect that the population of the Lagrangian solutions generated by the subgradient technique in the course of solving the Lagrangian dual is *sufficient* for the Algorithm to generate high quality feasible solutions. The quality of the dual bund is poor, improving for larger instances.

The feasible solution derived from the solution of the Lagrangian dual and corresponding to the last iteration of the subgradient technique not necessarily is the best feasible solution.

Moreover, typically $\varepsilon_{LF} > \varepsilon_{BF}$ and the best feasible solution is obtained on the early iterations of the subgradient method. Thus we may conclude that it is important to generate feasible solutions in all iterations of the subgradient technique.

5. Conclusions

Lagragian bound was presented for the two stage capacitated facility location problem. Two indicators were considered: the quality of the dual bound and the proximity of the Lagrangian based feasible solution. It turned out that relaxing the demand and the capacity plant constraints provides a rather poor dual bound, but the Lagrangian based feasible solutions are good, typically within 0.5-1.0% of the relative suboptimality. Relaxing the demand and the capacity plant constraints result in a decomposable Lagrangian problem with all sub problems analyzed by inspection. Thus this low cost relaxation seems to be promising to form the core of the Lagrangian based heuristics.

Solving the dual problem by the subgradient technique we compared two approaches to generate a feasible Lagrangian based solution. One is to get a feasible one by the solution of the dual problem, i.e. at the last iteration of the subgradient technique. Another approach is to generate feasible solutions in all iterations of the subgradient method and then choose the tightest. It turned out that the best (over all iterations) feasible solution never was obtained at the last iteration. That is, simply solving the Lagrangian dual and getting a corresponding Lagrangian based feasible solution is not sufficient to produce a tight feasible solution. On the contrary, the population of the Lagrangian solutions generated by the subgradient technique in the course of solving the Lagrangian dual is *sufficient* to generate high quality feasible solutions.

An interesting direction for the future research is improving the heuristic used to derive the feasible solutions. Some complements in this direction are in course.

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