A SUBGRADIENT PROJECTED ALGORITHM FOR NONCONVEX EQUILIBRIUM PROBLEMS

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RESUMO

Propomos um Método de Subgradiente de Plastria Projetado para resolver Problemas de Equilíbrio em Espaços Euclidianos. Assumimos que o segundo argumento da bifunção de equilíbrio é uma função quase-convexa e subdiferenciável inferiormente. Obtemos convergência Quase-Féjer para o conjunto solução e que toda sequência de iteradas converge para uma solução do problema.

PALAVRAS CHAVE. Problema de Equilíbrio, Método de Subgradiente Projetado, Subdiferencial de Plástria, Programação Matemática.

ABSTRACT

We propose a Projected Plastria's Subgradient Method for solving Equilibrium Problems in Euclidean Spaces. We assume that the second argument of the equilibrium bifunction is a quasiconvex and lower subdifferentiable function. We obtain quasi-Féjer convergence to the solution set and finally that the whole sequence of iterates converges to a solution of the problem.

KEYWORDS. Equilibrium Problem, Projected Subgradient Method, Plastria's Subdifferential, Mathematical Programming.

1. Introduction

Let C be a nonempty closed convex subset of $I\!R^n$ and let $f\colon I\!R^n \times I\!R^n \to (-\infty, +\infty]$ be a function such that $f(x,x) = 0$ for all $x \in C$ and $C \times C$ is contained in the effective domain of f. We consider the following *Equilibrium problem* (EP):

$$(EP)\begin{cases} \text{Find } \ x^* \in \ C \ \text{ such that} \\ f(x^*, y) \geq 0 \quad \forall \ y \ \in C. \end{cases} \tag{1}$$

In this paper we assume that the function $f(x,\cdot)\colon I\!R^n \longrightarrow (-\infty, +\infty]$ is essentially quasiconvex (see definition 4 below) and lower subdifferentiable at x, for all $x \in C$.

Equilibrium problems have been considered by several authors, see for example, Blum and Oettli (1994), Iusem and Sosa (2010), Konnov (2003), Lyashko et al (2011) and Santos and Scheimberg (2011a) and the references therein. It is well known that various classes of mathematical programming problems, variational inequalities, fixed point problems, Nash equilibrium in noncooperative games theory and minimax problems can be formulated in the form of (EP), see for instance, Blum and Oettli (1994).

Recently several numerical algorithms for solving the equilibrium problem have been proposed based on the subdifferential of the convex function $f(x,\cdot)$, see for example, Nguyen et al (2009), Santos and Scheimberg (2011b).

The paper is organized as follows: In Section 2 we recall useful basic notions. In Section 3 we define the algorithm and study its convergence.

2. Preliminaries

Let us start by introducing the definition of the so-called quasiconvex bifunction.

Definition 1 *A bifunction $f(x,\cdot) : C \subseteq \mathbb{R}^n \to (-\infty, +\infty]$ is said quasiconvex if for every $y, z \in C$ and for every $t \in [0,1]$ the following inequality holds:*

$$f\left(x, (1-t)y + tz\right) \leq \max\left\{f(x,y), f(x,z)\right\}. \tag{2}$$

In the following, we present the definition of the Plastria's subdifferential (see Plastria(1985)).

Definition 2 *Given a function $\varphi : \mathbb{R}^n \to \mathbb{R}$ and a point $x_0 \in \mathbb{R}^n$, the Plastria's lower subdifferential of φ at x_0 is defined and denoted by*

$$\partial^P \varphi(x_0) \;=\; \{\, \eta \in \mathbb{R}^n \;:\; \varphi(x) < \varphi(x_0) \Longrightarrow \langle \eta\,,\, x \,-\, x_0\rangle \;\leq\; \varphi(x) \,-\, \varphi(x_0)\,\}.$$

For next results, we use Gutiérrez's subdifferential, which is defined below.

Definition 3 *Given a function $\varphi : \mathbb{R}^n \to \mathbb{R}$ and a point $x_0 \in \mathbb{R}^n$, the Gutiérrez's lower subdifferential of φ at x_0 is defined and denoted by*

$$\partial^G \varphi(x_0) \;=\; \{\, \eta \in \mathbb{R}^n \;:\; \varphi(x) \leq \varphi(x_0) \Longrightarrow \langle \eta\,,\, x \,-\, x_0\rangle \;\leq\; \varphi(x) \,-\, \varphi(x_0)\,\}.$$

Next, we present a relationship between Plastria's subdifferential and Gutiérrez's subdifferential. For this, we consider the sets: $S_\varphi(x_0) = \{x \in C \;:\; \varphi(x) \leq \varphi(x_0)\}$ and $T_\varphi(x_0) = \{x \in C \;:\; \varphi(x) \;<\; \varphi(x_0)\}$.

Proposition 1 *Let $x_0 \in \mathbb{R}^n$. If the closure of $T_f(x_0)$ is equal to $S_f(x_0)$ then $\partial^P \varphi(x_0) = \partial^G \varphi(x_0)$.*

Proof: See Xu et al (1999). ∎

Definition 4 *A function $\varphi : \mathbb{R}^n \to \mathbb{R}$ quasiconvex is said to be essentially quasiconvex if each local minimizer is global.*

An important result about Plastria's subdifferential for essentially quasiconvex functions is presented below.

Proposition 2 *If $\varphi : \mathbb{R}^n \to \mathbb{R}$ is essentially quasiconvex continuous function then the closure of $T_\varphi(x_0)$ equals $S_\varphi(x_0)$ for all non-minimizer x_0 of φ. Consequently, $\partial^P\varphi(x_0) = \partial^G\varphi(x_0)$ for all non-minimizer x_0.*

Proof: See Da Cruz Neto(2012). ∎

Definition 5 *A function $\varphi : \mathbb{R}^n \to \mathbb{R}$ is said Lipschitz if there exists a number $0 \leq L < \infty$ such that*

$$|\varphi(x) - \varphi(y)| \leq L\,\|x - y\|, \quad \text{for all } x\,,\, y \in \mathbb{R}^n.$$

In the following theorem we have a known result of Plastria's subdifferential for the case of quasiconvex and Lipschitz continuous functions.

Theorem 1 *If $\varphi : \mathbb{R}^n \to \mathbb{R}$ is quasiconvex and Lipschitz continuous with constant L then the following holds:*

i) *$\partial^P\varphi(x_0) \neq \emptyset$ for every $x_0 \in \mathbb{R}^n$. Moreover, there exists $\eta \in \partial^P\varphi(x_0)$ com $\|\eta\| \leq L$.*

ii) *$\partial^P\varphi(x_0)$ is a closed and convex set for every $x_0 \in \mathbb{R}^n$;*

iii) *$0 \in \partial^P\varphi(x_0)$ if only if $x_0 \in \mathbb{R}^n$ is global minimizer of φ, in which case $\partial^P\varphi(x_0) = \mathbb{R}^n$.*

Proof: See Plastria(1985). ∎

We summarize in the next theorem the main result of the Plastria's subdifferential for the case of essentially quasiconvex and Lipschitz continuous functions.

Theorem 2 *Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be essentially quasiconvex and Lipschitz continuous with constant L then the following holds:*

i) *$\partial^G\varphi(x_0) \neq \emptyset$ for every $x_0 \in \mathbb{R}^n$.*

ii) *$\partial^G\varphi(x_0)$ is a closed and convex set for every $x_0 \in \mathbb{R}^n$;*

iii) *$0 \in \partial^G\varphi(x_0)$ if only if $x_0 \in \mathbb{R}^n$ is global minimizer of φ, in which case $\partial^G\varphi(x_0) = \{0\}$.*

Proof: See Xu et al (1999). ∎

In the following, we extend the definitions of Plastria's and Gutiérrez subdifferential for bifunctions.

Definition 6 *The Plastria's subdifferential of $f(x_0, \cdot)$ at $x_0 \in \mathbb{R}^n$ is defined and denoted by:*

$$\partial^P f(x_0, x_0) = \{\eta \in \mathbb{R}^n : f(x_0, y) < f(x_0, x_0) \Longrightarrow \langle \eta, y - x_0 \rangle \leq f(x_0, y) - f(x_0, x_0)\}$$

Definition 7 *The Gutiérrez's subdifferential of $f(x_0, \cdot)$ at $x_0 \in \mathbb{R}^n$ is defined and denoted by:*

$$\partial^G f(x_0, x_0) = \{\eta \in \mathbb{R}^n : f(x_0, y) \leq f(x_0, x_0) \Longrightarrow \langle \eta, y - x_0 \rangle \leq f(x_0, y) - f(x_0, x_0)\}$$

Note that when $x_0 \in C$, we have

$$\partial^G f(x_0, x_0) = \{\eta \in \mathbb{R}^n : f(x_0, y) \leq 0 \Longrightarrow \langle \eta, y - x_0 \rangle \leq f(x_0, y)\}.$$

From now on we denote the Plastria's subdifferential of f with respect to the second argument by $\partial_2^P f$.

Lemma 1 *A point $\overline{x} \in C$ is a solution of (EP) if and only if $0 \in \partial_2^P f(\overline{x}, \overline{x})$.*

Proof: Consider $\phi(x) = f(\overline{x}, x)$ and our conclusion follows from Theorem 9 in Censor and Segal (2006). ∎

We next recall the definition of quasi-Fejér sequence and an important related convergence theorem.

Definition 8 *A sequence $\{z^k\} \subset \mathbb{R}^n$ is called quasi-Fejér convergent to a set $U \subset \mathbb{R}^n$ if for every $u \in U$ there exists a sequence $\{\epsilon_k\} \subset \mathbb{R}_+$ such that*

$$\| z_{k+1} - u \|^2 \ \leq \ \| z_k - u \|^2 \ + \ \epsilon_k$$

with $\sum_{k=0}^{\infty} \epsilon_k < \infty$.

Proposition 3 *If $\{z_k\} \subset \mathbb{R}^n$ is a quasi-Fejér convergent sequence to a nonempty set U then $\{z_k\}$ is bounded. Further, if a cluster point $\overline{z}$ of $\{z_k\}$ belongs to U then $\lim_{k \to \infty} z_k = \overline{z}$.*

Proof: See Iusem et al (1994). ∎

3. The Algorithm and Its Convergence Analysis

3.1. The Projected Subgradient Method (PSM)

In this section, we defined the projected subgradient algorithm for solving equilibrium problem (EP). We show the well definedness of the generated sequence and we analyse its convergence.

In order to describe the (PSM) we assume that f is essentially quasiconvex and Lipschitz continuous function with constant L. Let $\{\alpha_k\}$, $\{\epsilon_k\}$ be sequences of nonnegative parameters such that

$$\alpha_k = \frac{\beta_k}{\gamma_k}, \quad \sum \beta_k = +\infty, \quad \sum \beta_k^2 < +\infty, \tag{3}$$

$$\gamma_k = \max\{\gamma, \|\eta^k\|\}, \quad \gamma > 0, \tag{4}$$

$$\sum \beta_k \epsilon_k < +\infty. \tag{5}$$

Step 0: Choose $x^0 \in C$. Set $k = 0$.

Step 1: Let $x^k \in C$. Obtain $\eta^k \in \partial_2^G f(x^k, x^k) \cap \overline{B(0, L)}$.
If $\eta^k = 0$, stop.

Step 2: Compute x^{k+1}:

$$x^{k+1} = \Pi_C \left[x^k - \alpha_k \eta^k \right]. \tag{6}$$

We recall that, given a nonempty closed and convex subset C of $\mathbb{R}^n$, the orthogonal projection of $x \in \mathbb{R}^n$ onto C, denoted by $\Pi_C(x)$, is the unique point in C, such that $\|\Pi_C(x) - y\| \leq \|x - y\|$ for all $y \in C$. Moreover, $\Pi_C(x)$ satisfies

$$\langle x - \Pi_C(x), z - \Pi_C(x)\rangle \leq 0 \;\; \forall\, y \in C. \tag{7}$$

We remark that all essentially quasiconvex function is, by definition, a quasiconvex function. Consequently $\partial^G f(x,x) \subset \partial^P f(x,x)$. In particular, quasiconvexity of f and Theorem 1 imply that there exists $\eta \in \partial^G f(x,x) \cap \overline{B(0,L)}$, where we mean by $\overline{B(0,L)}$ the set $\{x \in \mathbb{R}^n : \|x\| \leq L\}$.

3.2. Convergence Analysis

We consider the following additional assumptions in order to obtain convergence results.

A1. The solution set, S^*, of the problem (1) is nonempty;

A2. The bifunction f is pseudomonotone on C, that is, for all $x, y \in C$

$$f(x,y) \geq 0 \Longrightarrow f(y,x) \leq 0;$$

A3. If $x^*, \overline{x} \in C$, satisfy $f(\overline{x}, x^*) = f(x^*, \overline{x}) = 0$ then $x^* \in S^* \Rightarrow \overline{x} \in S^*$;

A4. $f(.,u)$ is upper semicontinuous for all $u \in C$.

Remark 1 *Assumptions A1 - A4 are usual requirements for (EP). A1 is a common assumption, see for example Konnov (2003), Muu and Quoc (2009). Assumption A2 is weaker than the monotonicity of f onto C, considered by several authors, see for instance Iusem and Sosa (2010). Assumption A3 is satisfied when the problem (EP) corresponds to an optimization problem, or when the problem (EP) is a reformulation of the variational inequality problem with a paramonotone operator. Assumption A3 can be recovered if the cyclic monotonicity of $-f$ is required (see for instance Bianchi et al (2005)). Assumption A4 is a common requirement for (EP), see for example Iusem and Sosa (2010), Nguyen et al (2009) and references therein.*

Lemma 2 *If the algorithm stops at step k, then x^k is a solution of (EP).*

Proof: In this case, we have that $\eta^k = 0$. Suppose, for the sake of contradiction, that $x^k \notin S^*$. Then, there exists $x \in C$ such that $f(x^k, x) < 0$. So, by definition of $\eta^k \in \partial_2^G f(x^k, x^k)$ we get that

$$0 = \langle \eta^k, x - x^k\rangle \leq f(x^k, x),$$

which is a contradiction. Hence, x^k is a solution of (EP). ■

From now on we denote by $\{x^k\}$ an infinite sequence generated by the Algorithm PSM. The next result will be useful in our convergence analysis.

Lemma 3 *Assume that A1 and A2 hold, then solution set of (EP), S^*, is a subset of U, where*

$$U = \{u \in C : \; f(x^k, u) \leq 0\}. \tag{8}$$

Proof: Let $x^* \in S^*$. Suppose, for the sake of contradiction, that $x^* \notin U$. Then, there exists $k \in \mathbb{N}$ such that $f(x^k, x^*) > 0$. By using A2, it results

$$f(x^k, x^*) > 0 \quad \Rightarrow \quad f(x^*, x^k) < 0,$$

which is a contradiction due to $f(x^*, x^k) \geq 0$ (Recall that x^* is a solution to the problem (EP)). ■

The next result is used to derive an important convergence property.

Proposition 4 *Assume A1 and let x^* be a solution of (EP). Then, for all $k \in \mathbb{N}$ it holds*

$$\|x^{k+1} - x^*\|^2 \leq \|x^k - x^*\|^2 + \beta_k^2 + 2\alpha_k f(x^k, x^*). \tag{9}$$

Proof: Let x^* be a solution of (EP), by Lemma 3 we know that $x^* \in U$ and

$$\begin{aligned} \|x^{k+1} - x^*\|^2 &= \|x^{k+1} - x^k\|^2 + \|x^k - x^*\|^2 + 2\langle x^{k+1} - x^k, x^k - x^*\rangle \\ &= \|x^{k+1} - x^k\|^2 + \|x^k - x^*\|^2 + 2\langle x^{k+1} - x^k + \alpha_k \eta^k, x^k - x^*\rangle \\ &\quad +2\langle \alpha_k \eta^k, x^* - x^k\rangle \\ &\leq \|x^{k+1} - x^k\|^2 + \|x^k - x^*\|^2 + 2\langle x^{k+1} - x^k + \alpha_k \eta^k, x^k - x^*\rangle \\ &\quad +2\alpha_k f(x^k, x^*), \end{aligned} \tag{10}$$

where the inequality is due to the definition of $\eta^k \in \partial_2^G f(x^k, x^k)$. Note that

$$\begin{aligned} \langle x^{k+1} - x^k + \alpha_k \eta^k, x^k - x^*\rangle &= \langle x^{k+1} - x^k + \alpha_k \eta^k, x^k - x^{k+1}\rangle \\ &\quad +\langle x^{k+1} - (x^k - \alpha_k \eta^k), x^{k+1} - x^*\rangle \\ &\leq \langle x^{k+1} - x^k + \alpha_k \eta^k, x^k - x^{k+1}\rangle \\ &= -\|x^{k+1} - x^k\|^2 + \alpha_k \langle \eta^k, x^k - x^{k+1}\rangle \\ &\leq -\|x^{k+1} - x^k\|^2 + \alpha_k \|\eta^k\| \|x^k - x^{k+1}\| \\ &\leq -\|x^{k+1} - x^k\|^2 + \beta_k \|x^k - x^{k+1}\|, \end{aligned} \tag{11}$$

where the first inequality is due to the definition of projection (see (7)) and the last inequality is obtained by (4) and (5). Then, by (10) and (11),

$$\begin{aligned} \|x^{k+1} - x^*\|^2 &\leq \|x^k - x^*\|^2 - \|x^{k+1} - x^k\|^2 + 2\beta_k \|x^{k+1} - x^k\| + 2\alpha_k f(x^k, x^*) \\ &\leq \|x^k - x^*\|^2 - \|x^{k+1} - x^k\|^2 + (\|x^{k+1} - x^k\|^2 + \beta_k^2) + 2\alpha_k f(x^k, x^*) \\ &= \|x^k - x^*\|^2 + \beta_k^2 + 2\alpha_k f(x^k, x^*). \end{aligned}$$

The proof is completed. ■

The result below is important for the convergence analysis.

Corollary 1 *Assume that A1, A2 and A3 are verified. Then the sequence $\{x^k\}$ is bounded.*

Proof: In fact, by using that $f(x^k, x^*) \leq 0$ it results from (9) that

$$\|x^{k+1} - x^*\|^2 \leq \|x^k - x^*\|^2 + \beta_k^2.$$

So, $\{x^k\}$ is Quasi-Féjer convergent to the solution set of (EP). Then, by Proposition 3, we close the proof of this corollary. ■

The next result will be useful to show that there exists a cluster point of $\{x^k\}$ belonging to S^*.

Theorem 3 *Suppose that A1 and A2 are hold. Then,*

$$\lim_{k\to+\infty} \sup f(x^k, x^*) = 0.$$

Proof: Firstly, we use the Proposition 4 to obtain that

$$\|x^{k+1} - x^*\|^2 \leq \|x^k - x^*\|^2 + 2\alpha_k f(x^k, x^*) + \beta_k^2. \tag{12}$$

So, by recalling that

$$\alpha_k f(x^k, x^*) \leq 0, \tag{13}$$

that is,

$$0 \leq 2\alpha_k[-f(x^k, x^*)] \leq \|x^k - x^*\|^2 - \|x^{k+1} - x^*\|^2 + \beta_k^2, \tag{14}$$

we obtain,

$$0 \leq \frac{\beta_k}{L}[-f(x^k, x^*)] \leq \|x^k - x^*\|^2 - \|x^{k+1} - x^*\|^2 + \beta_k^2. \tag{15}$$

Hence,

$$\begin{aligned} 0 &\leq L^{-1} \textstyle\sum_{k=0}^{m} \beta_k[-f(x^k, x^*)] \\ &\leq \|x^0 - x^*\|^2 - \|x^{m+1} - x^*\|^2 + \textstyle\sum_{k=0}^{m} \beta_k^2, \end{aligned} \tag{16}$$

by taking limits with $m \to +\infty$ we obtain

$$0 \leq \sum_{k=0}^{+\infty} \beta_k[-f(x^k, x^*)] \leq L\left[\|x^0 - x^*\|^2 + \sum_{k=0}^{+\infty} \beta_k^2\right] < +\infty. \tag{17}$$

We use the divergence of $\sum_{k=0}^{+\infty} \beta_k$ to conclude the proof, that is,

$$\lim_{k\to+\infty} \sup f(x^k, x^*) = 0.$$

■

Theorem 4 *Assume that A1-A4 are satisfied. Then, the whole sequence $\{x^k\}$ converges to a solution of (EP).*

Proof: Let $x^* \in S^*$. By Corollary 1, the sequence $\{x^k\}$ is bounded, then there exists a $\overline{x} \in C$ and a subsequence of $\{x^k\}$, namely $\{x^{k_j}\}$, such that $\lim_{j\to+\infty} x^{k_j} = \overline{x}$ and due to the Theorem 3 it follows that

$$\lim_{j\to+\infty} f(x^{k_j}, x^*) = 0. \tag{18}$$

From assumption A4 and Theorem 3, we obtain

$$\begin{aligned} f(\overline{x}, x^*) &\geq \limsup_{j\to+\infty} f(x^{k_j}, x^*) \\ &= \lim_{j\to+\infty} f(x^{k_j}, x^*) \\ &= 0. \end{aligned} \tag{19}$$

That is, $f(\overline{x}, x^*) \geq 0$. From assumption (A2) we have $f(x^*, \overline{x}) \leq 0$. Recall that $f(x^*, \overline{x}) \geq 0$ since x^* is a solution to (EP), it follows $f(x^*, \overline{x}) = 0$. Using the assumption (A2) again, we have $f(\overline{x}, x^*) = 0$. So,

$$f(\overline{x}, x^*) = 0, \;\; f(x^*, \overline{x}) = 0. \tag{20}$$

Hence, for $x^* \in S^*$ we obtain that (20) is satisfied and from (A3) we get that $\overline{x} \in S^*$. Then, by Proposition 3, the whole sequence $\{x^k\}$ is convergent to $\overline{x} \in S^*$. ■

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