COMPLETE COMMON NEIGHBORHOOD GRAPHS

Andréa Soares Bonifácio
Universidade Federal do Estado do Rio de Janeiro
Av. Pasteur, 458, CCET, Urca, Rio de Janeiro, Brasil
andreabonifacio@uniriotec.br

Romulo R. Rosa
Universidade Federal Fluminense - Rio das Ostras
Rua Recife, s/n, Jardim Bela Vista, Rio das Ostras, Brasil
romulorrosa@mat.uff.br

Ivan Gutman
University of Kragujevac
Faculty of Science P. O. Box 60, 34000 Kragujevac, Serbia
gutman@kg.ac.rs

Nair Maria Maia de Abreu
Universidade Federal do Rio de Janeiro
Av. Athos da Silva Ramos, 149, Centro de Tecnologia, Rio de Janeiro, RJ, Brasil
nair@pep.ufrj.br

RESUMO

O *grafo vizinhança comum* de um grafo G, ou simplesmente o *congrafo* de G, denotado por $con(G)$ tem sido tema de pesquisa recente por Alwardi *et. al.* (aceito para publicação, “in press”). Neste trabalho consideramos *congrafos iterados*, isto é, aqueles obtidos por meio de sucessivas aplicações do congrafo de um grafo dado e obtemos os seguintes resultados: provamos sobre condições bem gerais que o congrafo iterado é eventualmente um grafo completo. Caracterizamos os grafos com primeiro congrafo completo e derivamos uma fórmula para a matriz de adjacência de $con(G)$ em termos da matriz de adjacência de G.

PALAVRAS-CHAVE: Grafo Completo, Clique, Congrafo.

Área Principal: Teoria e Algoritimos em Grafos.

ABSTRACT

The *common neighborhood graph* of a graph G, or simply the *congraph* of G, denoted by $con(G)$ has been a theme of some recent research by Alwardi *et. al.* (accepted for publication, in press). In the present work we consider *iterated congraphs*, i.e., those obtained via successive application of the congraph of a given graph and we obtain the following results: we prove under very general conditions that the iterated congraph is eventually a complete graph. We characterize the graphs with complete congraphs and we derive a formula for the adjacency matrix of $con(G)$ in terms of the adjacency matrix of G.

KEYWORDS: Complete Graph, Clique, Congraph.

Main Area: Theory and Algorithms in Graphs.

1 Introduction

Let $G = G(V, E)$ be a graph of order n, $V = V(G)$ the vertex set and $E = E(G)$ the edge set of G. If $v_i, v_j \in V$ are linked to each other by an edge, we have $(v_i, v_j) \in E$ and we say that v_i and v_j are *adjacent* or *neighboring* vertices in G. The set of neighbors of v is denoted by $N_G(v)$ or simply $N(v)$. Two graphs G_1 and G_2 are *isomorphic*, denoted by $G_1 \cong G_2$, if there exists an one-to-one onto function $f : V(G_1) \longrightarrow V(G_2)$, such that $(u, v) \in E(G_1)$ if and only if $(f(u), f(v)) \in E(G_2)$. The *adjacency matrix* of G, $A(G) = (a_{ij})$, is the $n \times n$ matrix such that $a_{ij} = 1$ if the vertex v_i is adjacent to v_j, and $a_{ij} = 0$ otherwise. Adjacency matrices of isomorphic graphs are similar matrices. A *subgraph* $S(U, F)$ of G is a graph such that $U \subseteq V$ and $F \subseteq E$. An *induced subgraph* is a subgraph such that for all $v_i, v_j \in U$, if $(v_i, v_j) \in E$ then $(v_i, v_j) \in F$. The *complete* graph K_n is the graph with n vertices in which any two vertices are adjacent. A p*−clique* of a graph G is an induced subgraph of G isomorphic to K_p.

Definition 1.1. *We say that the common neighborhood graph of G or, simply, the congraph of G, denoted by con(G), is the graph with the same vertices as G such that (v_i, v_j) is an edge of $con(G)$ if and only if v_i and v_j have a common neighbor in G. In this case, we refer to G as the parent graph of $con(G)$.*

From this definition one can conclude that $N(v_i) \cap N(v_j) \neq \emptyset$ if and only if $(v_i, v_j) \in E(con(G))$.

The congraph of a graph G has been studied recently by Alwardi *et. al.* (to appear). In their work they obtain, among other results, the following theorem:

Theorem 1.2 (Alwardi *et. al. (to appear)*)**.** *The common neighborhood graph $con(G)$ is a connected graph if and only if the parent graph G is connected and non-bipartite.*

The same authors also observed that:

Remark 1.3. *Let G be a graph with n vertices.*

- *If G is a complete graph K_n then $con(K_n) \cong K_n$;*
- *If G is a cycle with odd length $G = C_{2k+1}$ then $con(C_{2k+1}) \cong C_{2k+1}$.*

From this remark, we have that if G has an odd cycle as a subgraph, then $con(G)$ contains an odd cycle isomorphic to that one in G. It's well known that a graph is bipartite if and only if it doesn't contain an odd cycle. Therefore, the congraph of a non-bipartite graph is also non-bipartite.

Definition 1.4. *We consider* iterated congraphs*, i.e., those obtained via successive applications of the congraph of a given graph. More formally, iterated congraphs are those obtained from a graph G as follows: $con^0(G) = G$ and $con^p(G) = con(con^{p-1}(G))$, for $p \in \mathbb{N}$.*

The paper is structured as follows: in Section 2, under certain conditions and for sufficiently large p, we prove that the iterated congraph, $con^p(G)$, is a complete graph. In Section 3, we characterize the graphs with complete congraphs. In Section 4 we derive a formula for the adjacency matrix of $con(G)$ in terms of the adjacency matrix of G.

2 Iterated congraphs

Our goal in this section is to prove that for some sufficiently large N, an iterated congraph of a connected non-bipartite graph that is not a cycle, is (isomorphic to) a complete graph.

2.1 Auxiliary Results

In the following lemmas we consider graphs for which we can explicitly put an upper bound on the number of iterates needed to obtain $con^p(G) \cong K_n$.

Let K_n^1 denote a graph obtained by attaching one vertex to the complete graph K_n and C_n^1 be the graph obtained by attaching one vertex to the cycle C_n. So, the attached vertex is a pendant vertex of G. In the Figure 1, we have the graphs $K_5^1, con(K_5^1)$ and $con^2(K_5^1)$.

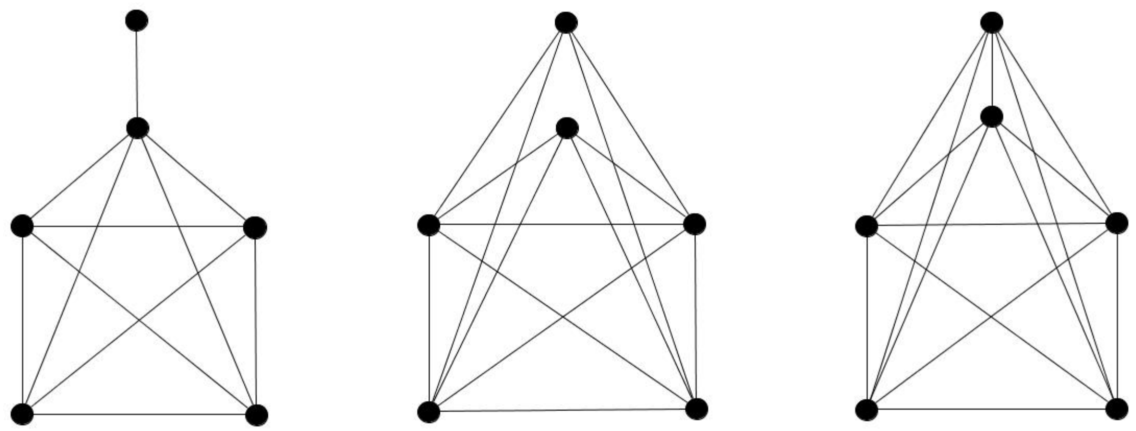

FIGURE 1. The graphs $K_5^1, con(K_5^1)$ and $con^2(K_5^1) \cong K_6$.

Lemma 2.1. *Let $n \geq 3$. If $G = K_n^1$ then $con^2(G) \cong K_{n+1}$.*

Proof. Let $n \geq 3$ and $G = K_n^1$. The graph K_n^1 is connected and contains the n-clique K_n. Without loss of generality, label the n vertices of K_n by $v_1, v_2, \ldots, v_n$ and take the pendant vertex v_* adjacent to v_1. From Remark 1.3, $con(K_n) \cong K_n$ and it is easy to see that $con(K_n) \subset con(K_n^1)$. Besides $con^2(K_n) \cong K_n \subset con^2(K_n^1)$. From this argument, every edge (v_i, v_j) on the n-clique K_n is an edge on $con^2(K_n^1)$.

Now take v_*. So, $N(v_*) = \{v_1\}$. For each $2 \leq i \leq n$, we have $N(v_*) \cap N(v_i) = \{v_1\}$ and then, for $2 \leq i \leq n$, (v_i, v_*) is an edge of $con(K_n^1)$. Moreover, $N(v_*) \cap N(v_1) = \{v_1\} \cap N(v_1) = \emptyset$. Consequently, (v_1, v_*) is not an edge of $con(K_n^1)$.

Let v be a vertex of K_n^1 and denote $N_1(v)$, the set of vertices adjacent to v in $con(K_n^1)$, the congraph of K_n^1. Then we have, $N_1(v_*) = V(K_n) - \{v_1\}$ and $N_1(v_1) = V(K_n) - \{v_*, v_1\}$. So, $N_1(v_*) \cap N_1(v_1) \neq \emptyset$ and (v_*, v_1) is an edge on $con^2(K_n^1)$. For $i = 2, \ldots, n$, we obtain, $N_1(v_i) = V(K_n^1) - \{v_i\}$ and, then, $N_1(v_*) \cap N_1(v_i) \neq \emptyset$. Consequently, (v_*, v_i) is an edge on $con^2(K_n^1)$. As every $(v_i, v_j) \neq (v_*, v_i)$ is an edge in $con^2(K_n^1)$, we get $con^2(K_n^1) \cong K_{n+1}$. □

The next lemma is slightly technical but is very important for the proof of our main result.

Lemma 2.2. *For $0 \leq p \leq 2k-1$, we have $K_{p+3} \subseteq con^{2p+1}(C_{2k+1}^1)$.*

Proof. Consider $G = C_{2k+1}^1$. Let $v_0, v_1, \cdots, v_{2k}$ be the labels of the vertices of C_{2k+1} in G and v_* the pendant vertex of G attached to v_0. So, $N(v_*) = \{v_0\}$. Therefore $N(v_*) \cap N(v_1) = N(v_*) \cap N(v_{2k}) = N(v_1) \cap N(v_{2k}) = \{v_0\}$. Then, the edges $(v_*, v_1), (v_1, v_{2k})$ and $(v_{2k}, v_*) \in E(con(C_{2k+1}^1)$ and v_1, v_{2k} and v_* constitutes a 3-clique in $con(C_{2k+1}^1)$. This verifies the claim for $p = 0$.

Now let $p \in \mathbb{N}$, $1 \leq p \leq 2k-2$, and assume the induction hypothesis for p:

$$K_{p+3} \subseteq con^{2p+1}(C_{2k+1}^1).$$

We wish to prove that

$$K_{(p+1)+3} \subseteq con^{2(p+1)+1}(C^1_{2k+1}).$$

If $p \leq 2k-2$ then $p+3 \leq 2k+1$. Then, $K_{p+3} \subseteq con^{2p+1}(C^1_{2k+1})$ where K_{p+3} has at most $2k+1$ vertices. Consequently, there is a vertex v_{i_0} of $con^{2p+1}(C^1_{2k+1})$ that is not a vertex of K_{p+3}. Since $con^{2p+1}(C^1_{2k+1})$ is a connected graph, v_{i_0} is attached to K_{p+3}. We call this subgraph K^1_{p+3}. From Lemma 2.1, we will have $K_{(p+3)+1} \subseteq con^{2p+1+2}(C^1_{2k+1})$ or, equivalently, $K_{(p+1)+3} \subseteq con^{2(p+1)+1}(C^1_{2k+1})$. □

The Figure 2 shows the graph C^1_5 and its four first iterated congraphs.

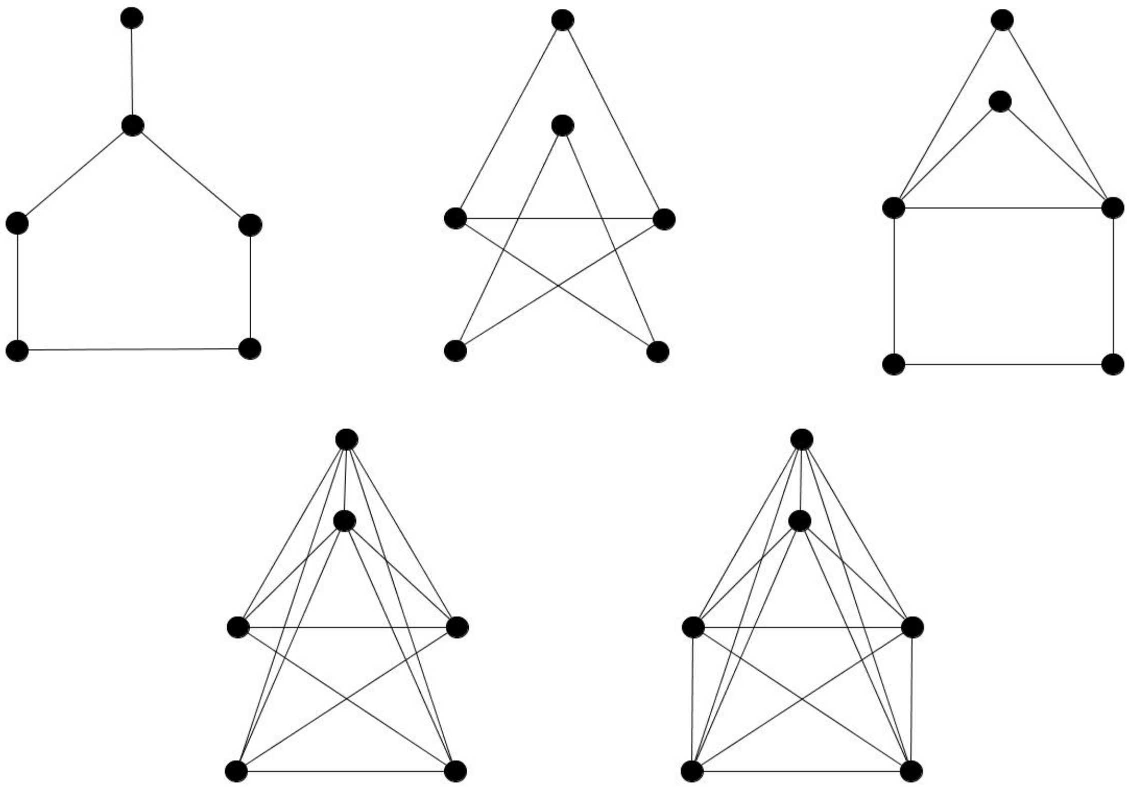

FIGURE 2. The graph C^1_5 and its iterated congraphs: $con(C^1_5), con^2(C^1_5), con^3(C^1_5)$ and $con^4(C^1_5) \cong K_6$.

In the next lemma, we calculate an upper bound on the number of iterates needed to obtain a complete congraph starting from C^1_{2k+1}.

Lemma 2.3. *If $G = C^1_{2k+1}$, there is $N' \leq 4k-1$ such that $con^{N'}(G) \cong K_{2k+2}$.*

Proof. Let $p = 2k-1$. From Lemma 2.2, we have

$$K_{2k-1+3} \subseteq con^{2(2k-1)+1}(C^1_{2k+1}).$$

So,

$$K_{2k+2} \subseteq con^{4k-1}(C^1_{2k+1}).$$

As the congraph of G has the same number of vertices of G, we get

$$con^{4k-1}(C^1_{2k+1}) \cong K_{2k+2}.$$

□

2.2 Proof of the main Theorem

First, we want to check a sufficient condition for G to contain a C^1_{2k+1}, an this is in the following remark:

Remark 2.4. *If G is connected, $C_{2q+1} \subseteq G$ and $G \neq C_{2q+1}$ then $\exists\, k$, $1 \leq k \leq q$, such that $C^1_{2k+1} \subseteq G$.*

Proof. Let v_* be a vertex of G that is not a vertex of C_{2q+1}. Since G is connected, the result holds. Otherwise, G has an edge that is a chord in C_{2q+1}. Since $2q+1$ is odd, this chord divides C_{2q+1} in two cycles of distinct parity. Therefore, there is k, $1 \leq k < q$ such that $C_{2k+1} \subseteq G$. This way, there is at least one vertex v_* in the cycle C_{2q+1} of G that is not a vertex of C_{2k+1} and it is attached to C_{2k+1}. So, the remark follows. □

We are now ready to prove our main result.

Theorem 2.5. *Let G be a graph with $n > 3$ vertices. The existence of an $N \in \mathbb{N}$ such that $con^N(G)$ is complete is equivalent to the following: G is a connected non-bipartite graph and is not a cycle.*

Proof. ($\Rightarrow$) If for some $N \in \mathbb{N}$ we have that $con^N(G)$ is complete then by the Theorem 1.2 we have that $con^{(N-1)}(G)$ is connected and non-bipartite. Applying this same argument $N-1$ times, we have that G is connected and non-bipartite graph and, therefore, it contains an odd cycle. By hypothesis, $con^N(G)$ is complete. By the Remark 1.3 we know that $C_{2k+1} \cong con(C_{2k+1}) \cong con^N(C_{2k+1})$. Then G cannot be an odd cycle (because an odd cycle is not a complete graph for $n > 3$).

($\Leftarrow$) Let G be a connected non-bipartite graph with n vertices, $G \neq C_n$. From Remark 2.4, G has a C^1_{2k+1} as a subgraph. From Lemma 2.3, there is $N' \leq 4k-1$ such that $K_{2k+2} \subseteq con^{N'}(G)$. If $2k+2 = n$ we are done. Otherwise, there is a pendant vertex in G which is attached to K_{2k+2}. From Lemma 2.1, $con^N(G) \cong K_n$, where $N = N' + 2(n-(2k+2)) \leq 4k-1+2(n-(2k+2)) = 2n+3$. □

3 Graphs with complete congraphs

This section is devoted to prove a theorem that characterizes the class of graphs whose congraph (the first iterated congraph) is a complete graph. However, in order to get there, we need to establish, in the next lemma, a necessary condition to have such graphs.

Lemma 3.1. *Let G be a graph such that the congraph of G is a complete graph. Then, every edge of G is in a 3-clique of G.*

Proof. A necessary condition for the congraph of G be a complete graph is that every edge of G has to be an edge also of the congraph of G. Let G be a graph such that $con(G)$ is complete. Let (v_i, v_j) be an edge of G. It follows from the hypothesis that (v_i, v_j) is also an edge of $con(G)$. That means the vertices v_i and v_j have at least one common neighbor v_k in G. Then, (v_i, v_k) and (v_k, v_j) are edges on G and, consequently, v_i, v_j, v_k is a 3-clique in G. □

Theorem 3.2. *Let G be a connected graph. Then $con(G)$ is complete if and only if for any given pair of vertices v_i, v_j of G there is a 3-clique containing v_i and a 3-clique containing v_j such that those cliques have a common vertex.*

Proof. ($\Rightarrow$) Suppose $con(G)$ is complete. Take a pair of vertices v_i, v_j on G. As $con(G)$ is complete there is a common neighbor v_k on G for v_i and v_j. From Lemma 3.1, the edge (v_i, v_k) is on a 3-clique in G and also (v_j, v_k) is on a 3-clique in G. So, those 3-cliques have v_k in common.

($\Leftarrow$) Let G be a connected graph satisfying the conditions in the above statement. Let v_i, v_j be two vertices of G. If they are on a same 3-clique on G then (v_i, v_j) is an edge in $con(G)$.

Now consider v_i, v_j on different 3-cliques with a common vertex v_k. This way v_k is a common neighbor for v_i and v_j and therefore (v_i, v_j) is an edge on $con(G)$. So we have proved that, if G satisfies the above condition, then for any pair of vertices v_i, v_j we have that (v_i, v_j) is an edge of $con(G)$ and therefore $con(G)$ is complete. □

In the Figure 3 below we have some graphs whose first congraph is a complete graph.

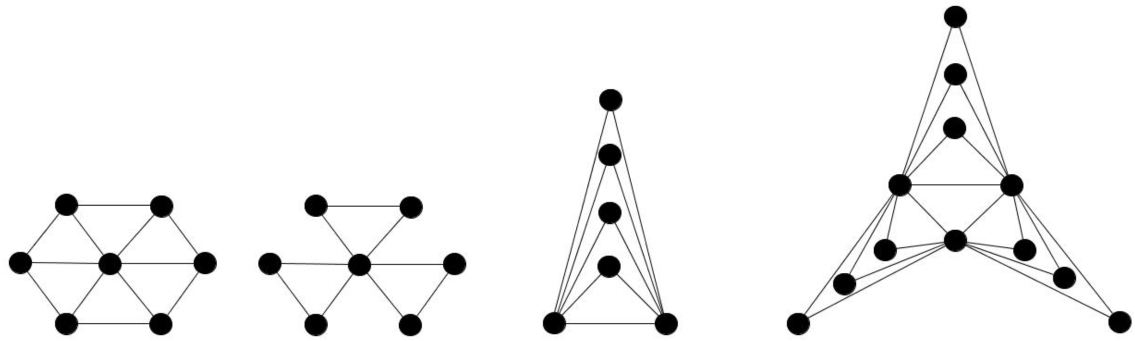

FIGURE 3. Some graphs whose congraph is complete.

Remark 3.3. *In a complete graph one can guarantee that each vertex is connected to all the other vertices by a path of length one. The class of graphs that have a complete congraph is exactly the class of connected non-bipartite graphs where each vertex is connected to all the other vertices by a path of length two.*

4 The adjacency matrix of con(G)

In this section we obtain a simple formula for the adjacency matrix of $con(G)$ in terms of the adjacency matrix of G. Let $A = A(G)$ be the adjacency matrix of a connected graph G and denote by b_{ij} the elements of A^2 . So,

$$b_{ij} := \sum_{k=1}^{n} a_{ik}a_{kj}.$$

Let $\hat{A} = A(con(G))$ be the adjacency matrix of the graph $con(G)$ and denote by J the square matrix of the same order of A with all entries equal to 1.

Remark 4.1. *For* $x \in \mathbb{R}$, *we have* $\frac{x+1-|x-1|}{2} = 1 \Leftrightarrow x \geq 1$ *and* $\frac{x+1-|x-1|}{2} = 0 \Leftrightarrow x = 0.$

Theorem 4.2. *For every connected graph G, we have the following formula:*

$$\hat{A} = \frac{1}{2}\left(A^2 + J - |A^2 - J|\right) - I.$$

Proof. Consider $i \neq j$. An element b_{ij} of A^2 is the number of paths of length two linking the vertices v_i and v_j in G. So, we have that $\hat{a}_{ij} = 1 \Leftrightarrow b_{ij} \geq 1$. It follows immediately that $\hat{a}_{ij} = 0 \Leftrightarrow b_{ij} = 0$.

From Remark 4.1, we have, for $i \neq j$, the following formula for $\hat{a}_{ij}$:

$$\hat{a}_{ij} = \frac{b_{ij} + 1 - |b_{ij} - 1|}{2}. \tag{4.1}$$

Now we consider a diagonal element b_{ii}. As G is connected, there is $k \neq i$ such that $a_{ik} = 1$. By the symmetry of A , $a_{ik} = a_{ki}$ and then

$$b_{ii} := \sum_{k=1}^{n} a_{ik}a_{ki} \geq 1.$$

The matrix $\hat{A}$ is an adjacency matrix and therefore $\hat{a}_{ii} = 0$. This way we have the following formula for $\hat{a}_{ii}$:

$$\hat{a}_{ii} = \frac{b_{ii} + 1 - |b_{ii} - 1|}{2} - 1. \tag{4.2}$$

From the equations 4.1 and 4.2 the result follows. □

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