CLAIO SBPO
Congreso Latino-Iberoamericano de Investigación Operativa
Simpósio Brasileiro de Pesquisa Operacional
September 24-28, 2012
Rio de Janeiro, Brazil

# Nonrepetitive, acyclic and clique colorings of graphs with few $P_4$'s

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**Abstract**

In this paper, we propose algorithms to determine the Thue chromatic number and the clique chromatic number of $P_4$-tidy graphs and $(q, q-4)$-graphs. These classes include cographs and $P_4$-sparse graphs. All algorithms have linear-time complexity, for fixed $q$, and then are fixed parameter tractable. All these coloring problems are known to be NP-hard for general graphs. We also prove that every connected $(q, q-4)$-graph with at least $q$ vertices is 2-clique-colorable and that every acyclic coloring of a cograph is also nonrepetitive, generalizing a result from [28]. Finally, we show that the algorithm from [31] can also be used to compute the acyclic chromatic number of distance hereditary graphs and graphs with a given split decomposition tree with bounded width.

**KEYWORDS: nonrepetitive coloring, Thue chromatic number, clique coloring, acyclic coloring, graphs with few $P_4$'s, fixed parameter tractability, linear time algorithms, primeval decompostion, split decomposition, distance hereditary graphs.**

# 1 Introduction

The graph terminology used here follows [10]. The complete bipartite graph with partitions of size $m$ and $n$ is denoted by $K_{m,n}$. A star is a $K_{1,n}$. A $P_4$ is an induced path with four vertices. A cograph is any $P_4$-free graph.

A $k$-coloring of $G$ is a partition $\{V_1, \ldots, V_k\}$ of $V(G)$. A proper $k$-coloring is a $k$-coloring such that every color class induces a stable set. The chromatic number $\chi(G)$ of $G$ is the smallest integer $k$ such that $G$ admits a proper $k$-coloring.

An *acyclic* coloring is a proper coloring such that every pair of color classes induces a forest. A *star* coloring is a proper coloring such that every pair of color classes induces a forest of stars. A *nonrepetitive* coloring is a proper coloring such that no path has an $xx$ pattern of colors, where $x$ is a sequence of colors. A *harmonious* coloring is a proper coloring such that every pair of color classes induces at most one edge.

In 2011, it was proved that every acyclic coloring of a cograph is also a star coloring [28]. In this paper, we prove that it is also nonrepetitive.

The acyclic, star, Thue and harmonious chromatic numbers of $G$, denoted respectively by $\chi_a(G)$, $\chi_{st}(G)$, $\pi(G)$, $\chi_h(G)$, are the minimum number of colors $k$ such that $G$ admits an acyclic, star, nonrepetitive and harmonious coloring with $k$ colors. By the definitions, it is easy to see that

$$\chi(G) \;\leq\; \chi_a(G) \;\leq\; \chi_{st}(G) \;\leq\; \pi(G) \;\leq\; \chi_h(G).$$

Determining the acyclic chromatic number is NP-Hard even for bipartite graphs [16] and deciding if $\chi_a(G) \leq 3$ is NP-Complete [26]. In 2004, Albertson et al. [2] proved that computing the star chromatic number is NP-hard even for planar bipartite graphs. In 2007, Asdre et al. [3] proved that determining the harmonious chromatic number is NP-hard for interval graphs, permutation graphs and split graphs.

Borodin proved that $\chi_a(G) \leq 5$ for every planar graph $G$ [11]. In 2004, Fertin, Raspaud and Reed give exact values of $\chi_{st}(G)$ for several graph classes [18]. In 2004, Campbell and Edwards [12] obtained new lower bounds for $\chi_h(G)$ in terms of the independence number.

In 2002, Alon et al. [1] proved a relation between the $\pi(G)$ and $\Delta(G)$. In 2008, Barát and Wood [8] proved that every graph $G$ with treewidth $t$ and maximum degree $\Delta$ satisfies $\pi(G) = O(k\Delta)$ (it was also proved that $\pi(G) \leq 4^t$ [21]). In 2009, Marx and Schaefer [29] proved that determining whether a particular coloring of a graph is nonrepetitive is coNP-hard, even if the number of colors is limited to four. In 2010, Grytczuk et al. [23] investigated list colorings which are nonrepetitive and proved that the Thue choice number of $P_n$ is at most 4 for every $n$. See [21] and [22] for a survey on nonrepetitive colorings.

A *clique* coloring is a coloring (not necessarily a proper coloring) such that every maximal clique receives at least two colors. The clique chromatic number $\chi_c(G)$ is the minimum number $k$ such that $G$ has a clique coloring with $k$ colors.

In 2004, Bacsó et al. [7] proved several results for 2-clique-colorable graphs. In 2002, Kratochvíl and Tuza [27] proved that determining the clique-chromatic number is polynomial time solvable for planar graphs, but is NP-Hard for perfect graphs.

Many NP-hard problems were proved to be polynomial time solvable for cographs. For example, Lyons [28] obtained in 2011 a polynomial time algorithm to find an optimal acyclic and an optimal star coloring of a cograph. However, it is known that computing the harmonious chromatic number of a disconnected cograph is NP-hard [9].

Some superclasses of cographs, defined in terms of the number and structure of its induced $P_4$'s, can be completely characterized by their primeval decomposition. Among these classes, we cite $P_4$-sparse graphs, $P_4$-lite graphs, $P_4$-tidy graphs and $(q, q-4)$-graphs.

Babel and Olariu [5] defined a graph as $(q, q-4)$-*graph* if no set of at most $q$ vertices induces more than $q-4$ induced $P_4$'s. Cographs and $P_4$-sparse graphs are precisely $(4, 0)$-graphs and $(5, 1)$-graphs respectively. $P_4$-lite graphs are special $(7, 3)$-graphs. We say that a graph is $P_4$-tidy if, for every $P_4$ induced by $\{u, v, x, y\}$, there exists at most one vertex $z$ such that $\{u, v, x, y, z\}$ induces more than one $P_4$. Since the complement of a $P_4$ is also a $P_4$, these graph classes are self-complementary.

In this paper, we prove the following main results:

**Theorem 1.1.** *Let $q$ be a fixed integer and let $G$ be a $P_4$-tidy or a $(q, q-4)$-graph. There exist linear time algorithms to obtain the Thue chromatic number and the clique chromatic number of $G$.*

These algorithms are dynamic programming, obtained from a graph decomposition, called primeval decomposition, described in the next section, and obtained from the lemmas of the Sections 3 and 4.

Let $q(G)$ be the minimum integer $q$ such that $G$ is a $(q, q-4)$-graph. Theorem 1.1 proves that the nonrepetitive and the clique coloring problems are fixed parameter tractable on the parameter $q(G)$. Recently, Campos et al. [14] proved that the cochromatic number and the cocoloring problem are fixed parameter tractable on $q(G)$. Also in 2011, Campos et al. proved the same for the acyclic coloring, the star coloring and the harmonious coloring problems [13].

The theorem below generalizes a result from [28].

**Theorem 1.2.** *Every acyclic coloring of a cograph is also nonrepetitive. Moreover, every connected $(q, q-4)$-graph with at least $q$ vertices is 2-clique-colorable.*

Finally, we also prove that the algorithm from [31] to obtain an optimal coloring of a graph using its split decomposition can also be used to obtain an optimal acylic coloring of such graphs. Distance hereditary graphs are graphs in which the distances in any connected induced subgraph are the same as they are in the original graph. Cographs are distance hereditary. With this, we have our last main result.

**Theorem 1.3.** *There exist polynomial time algorithms to obtain an optimal acyclic coloring of distance hereditary graphs and graphs with a given split decomposition with bounded width.*

The split decomposition is explained in Section 5. The proofs of these theorems follow directly from the lemmas of the next sections.

## 2 Primeval and Modular decompositions

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two vertex disjoint graphs. The disjoint union of $G_1$ and $G_2$ is the graph $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$. The join is the graph $G_1 \vee G_2 = (V_1 \cup V_2, E_1 \cup E_2 \cup \{uv : u \in V_1, v \in V_2\})$.

A *spider* is a graph whose vertex set has a partition $(R, C, S)$, where $C = \{c_1, \ldots, c_k\}$ and $S = \{s_1, \ldots, s_k\}$ for $k \geq 2$ are respectively a clique and a stable set; $s_i$ is adjacent to $c_j$ if and only if $i = j$ (a thin spider), or $s_i$ is adjacent to $c_j$ if and only if $i \neq j$ (a thick spider); and every vertex of $R$ is adjacent to each vertex of $C$ and non-adjacent to each vertex of $S$.

Jamison and Olariu [5] proved an important structural theorem for $(q, q-4)$-graphs, using their primeval decomposition, which can be obtained in linear time. A graph is *p-connected* if, for every bipartition of the vertex set, there is a crossing $P_4$. A *separable p-component* is a maximal p-connected subgraph with a particular bipartition $(H_1, H_2)$ such that every crossing $P_4$ $wxyz$ satisfies $x, y \in H_1$ and $w, z \in H_2$.

**Theorem 2.1** (Characterizing $(q, q-4)$-graphs [5])**.** *A graph $G$ is a $(q, q-4)$-graph if and only if exactly one of the following holds:*

*(a) $G$ is the union or the join of two $(q, q-4)$-graphs;*

*(b) $G$ is a spider $(R, C, S)$ and $G[R]$ is a $(q, q-4)$-graph;*

*(c) $G$ contains a separable p-component $H$, with bipartition $(H_1, H_2)$ and $|V(H)| \leq q$, such that $G - H$ is a $(q, q-4)$-graph and every vertex of $G - H$ is adjacent to every vertex of $H_1$ and non-adjacent to every vertex of $H_2$;*

*(d) $G$ has at most $q$ vertices or $V(G) = \emptyset$.*

Using the modular decomposition of $P_4$-tidy graphs, Giakoumakis et al. proved a similar result for this class [20]. A *quasi-spider* is a graph obtained from a spider $(R, C, S)$ by replacing at most one vertex from $C \cup S$ by a $K_2$ (the complete graph on two vertices) or a $\overline{K_2}$ (the complement of $K_2$).

**Theorem 2.2** (Characterizing $P_4$-tidy graphs [20])**.** *A graph $G$ is a $P_4$-tidy graph if and only if exactly one of the following holds:*

*(a) $G$ is the union or the join of two $P_4$-tidy graphs;*

*(b) $G$ is a quasi-spider $(R, C, S)$ and $G[R]$ is a $P_4$-tidy graph;*

*(c) $G$ is isomorphic to $P_5$, $\overline{P_5}$, $C_5$, $K_1$ or $V(G) = \emptyset$.*

As a consequence, a $(q, q-4)$-graph (resp. a $P_4$-tidy graph) $G$ can be decomposed by successively applying Theorem 2.1 (resp. Theorem 2.2) as follows: If (a) holds, apply the theorem to each component of $G$ or $\overline{G}$ (operations disjoint union and join). If (b) holds, apply the theorem to $G[R]$ (operation spider or quasi-spider). Finally, if (c) holds and $G$ is a $(q, q-4)$-graph, then apply the theorem to $G - H$ (operation small subgraph).

It was also proved in [5] that every p-connected $(q, q-4)$-graph with $q \geq 8$ has at most $q$ vertices. With this, we can obtain $q(G)$ in $O(n^7)$ time for every graph $G$ from its primeval decomposition (observe that $q(G)$ can be greater than $n$ and, if this is the case, $q(G)$ is the number of induced $P_4$'s of $G$ plus four).

The idea now is to consider the graph by the means of its decomposition tree obtained as described. According to the coloring parameter to be determined, the tree will be visited in an up way or bottom way fashion. We notice that the primeval and modular decomposition of any graph can be obtained in linear time [5].

## 3 Disjoint Union, Join and Spiders

We start by recalling a result from [28] for the acyclic chromatic numbers.

**Lemma 3.1** ($\chi_a$ for union and join [28]). *Given graphs $G_1$ and $G_2$ with $n_1$ and $n_2$ vertices res-pectively:*

$$\chi_a(G_1 \cup G_2) = \max\{\chi_a(G_1), \chi_a(G_2)\},$$
$$\chi_a(G_1 \vee G_2) = \min\{\chi_a(G_1) + n_2, \chi_a(G_2) + n_1\},$$

The next lemma shows how to obtain the Thue chromatic number for union and join operations. It is not too difficult to see that Lemmas 3.1 and 3.2 implies that, if $G$ is a cograph, then $\pi(G) = \chi_a(G)$ and every acyclic coloring of a cograph is also nonrepetitive.

**Lemma 3.2** ($\pi(G)$ for union and join). *Given graphs $G_1$ and $G_2$ with $n_1$ and $n_2$ vertices respectively:*

$$\pi(G_1 \cup G_2) = \max\{\pi(G_1), \pi(G_2)\},$$
$$\pi(G_1 \vee G_2) = \min\{\pi(G_1) + n_2, \pi(G_2) + n_1\}.$$

The two following lemmas deal with spiders and quasi-spiders and are proved in Section 6. We will consider $\pi(G[R]) = 0$ whenever $R = \emptyset$.

**Lemma 3.3** ($\pi(G)$ for spiders). *Let $G$ be a spider $(R, C, S)$, where $|C| = |S| = k$. Then $\pi(G) = \pi(G[R]) + k$, unless $R = \emptyset$ and $G$ is thick, when in this case, $\pi(G) = k + 1$.*

**Lemma 3.4** ($\pi(G)$ for quasi-spiders). *Let $G$ be a quasi-spider $(R, C, S)$ such that* $\min\{|C|, |S|\} =$ *$k$ and* $\max\{|C|, |S|\} = k + 1$. *Let $H = K_2$ or $H = \overline{K_2}$ be the subgraph that replaced a vertex of $C \cup S$. Then*

$$\pi(G) = \begin{cases} \pi(G[R]) + k, & \text{if } H \in S \text{ and } G \text{ is thin,} \\ \pi(G[R]) + k, & \text{if } H \in S,\ G \text{ is thick} \\ & \qquad\qquad \text{and } R \neq \emptyset, \\ \pi(G[R]) + k + 2, & \text{if } H \in C,\ G \text{ is thick} \\ & \qquad\qquad \text{and } R = \emptyset, \\ \pi(G[R]) + k + 1, & \text{otherwise.} \end{cases}$$

Next lemma deals with the clique chromatic number.

**Lemma 3.5** ($\chi_c$ for union, join and quasi-spiders). *Let $G_1$ and $G_2$ be two graphs. Then, $\chi_c(G_1 \cup G_2) = \max\{\chi_c(G_1), \chi_c(G_2)\}$ and $\chi_c(G_1 \vee G_2) = 2$. If $G$ is a quasi-spider, then $\chi_c(G) = 2$.*

## 4 Coloring $(q, q-4)$-graphs

In this section, suppose that $G$ is a $(q, q-4)$-graph which contains a separable $p$-component $H$, with bipartition $(H_1, H_2)$ and at most $q$ vertices, such that every vertex from $G - H$ is adjacent to all vertices in $H_1$ and non-adjacent to all vertices in $H_2$. Let $n'$ be the number of vertices of $G - H$. If $G - H$ is empty, consider $\chi_a(G - H) = \chi_{st}(G - H) = \pi(G - H) = 0$. Given a coloring $\psi$ of $H$, let $k(\psi)$ be the number of colors of $\psi$.

Theorems below prove that determining the chromatic numbers $\pi$ and $\chi_c$ for item (c) of Theorem 2.1 is linear time solvable, if $q$ is a fixed integer.

**Lemma 4.1.** *Given a coloring $\psi$ of $H$, let $k_2(\psi)$ be the number of colors with no vertex of $H_1$ and with no vertex of $H_2$ which is neighbor of two vertices from $H_1$ with the same color. Then*

$$\pi(G) = \min\Big\{ \min_{\psi\in C_\pi(H)} \Big\{k(\psi) + \max\{0, n' - k_2(\psi)\}\Big\},$$

$$\min_{\psi'\in C'_\pi(H)} \Big\{k(\psi') + \max\{0, \pi(G-H) - k_2(\psi')\}\Big\}\Big\}$$

*where $C_\pi(H)$ is the set of all nonrepetitive colorings of $H$, and $C'_\pi(H) \subseteq C_\pi(H)$ is the subset of nonrepetitive colorings such that all vertices from $H_1$ receive distinct colors.*

**Lemma 4.2.** *If $G - H$ is not empty, then $\chi_c(G) = 2$ (coloring the vertices of $G - H$ and $H_2$ with the color 1 and the vertices of $H_1$ with the color 2). If $G - H$ is empty, then $G$ has less than $q$ vertices and*

$$\chi_c(G) = \min_{\psi\in C_c(H)} \Big\{k(\psi)\Big\},$$

*where $C_c(H)$ is the set of all clique-colorings of $H$.*

**Theorem 4.3.** *If $G$ is a $P_4$-tidy or $(q, q-4)$-graph, then we can obtain a minimum nonrepetitive and a minimum clique coloring of $G$ and determine $\pi(G)$ and $\chi_c(G)$ in linear time.*

*Proof.* From Section 2, we can obtain the primeval decomposition in linear time. From lemmas of Sections 3 and 4, we are finished. □

## 5 Split decomposition

A split of a graph $G$ is a partition of $V(G)$ into two sets $V_1$ and $V_2$ with at least two vertices such that every vertex in $V_1$ with a neighbor in $V_2$ has the same neighborhood in $V_2$. Given a split $(V_1, V_2)$ of a graph $G$, we can decompose $G$ into $G_1$ and $G_2$, where, for $i \in \{1, 2\}$, $G_i$ is the subgraph of $G$ induced by $V_i$ with an additional vertex $u$, called a *marker*, such that the neighborhood of $v$ in $G_i$ is the set of those vertices in $V_i$ which are adjacent to a vertex outside of $V_i$. A graph is prime if it does not have a split.

Given graphs $G_1$ and $G_2$ such that $V(G_1) \cap V(G_2) = \{v\}$, let $G_1 * G_2$ be the graph with vertex set $(V_1 \cup V_2) \setminus \{v\}$, and edge set $\{xy \in E(G_1) : x \neq v \text{ and } y \neq v\} \cup \{xy \in E(G_2) : x \neq v \text{ and } y \neq v\} \cup \{xy : x \in N_{G_1}(v) \text{ and } y \in N_{G_2}(v)\}$. Clearly, if $G$ is decomposable into $G_1$ and $G_2$, then $G = G_1 * G_2$.

The split decomposition of a graph is the recursive decomposition of the graph using simple decompositions in splits until none of the obtained graphs can be decomposed further. The split decomposition tree of the graph $G$ is the tree $T$ in which each node $h$ corresponds to a prime graph denoted by $G^*_h$ obtained by the split decomposition. Furthermore two nodes $h$ and $h'$ of $T$ are adjacent if and only if the corresponding graphs $G^*_h$ and $G^*_{h'}$ have a common marker. If $h'$ is the parent of the node $h$ in $T$, let $v_h$ be the unique marker belonging to $G^*_h$ and $G^*_{h'}$, which we call parent marker. Let $G_h$ be the graph corresponding to the subtree of $T$ rooted at the node $h$.

The split decomposition of a graph is not necessarily unique. In [17], Dahlhaus obtained a linear time algorithm to compute a split decomposition of a graph. See [15] for more details. Distance hereditary graphs are completely decomposable by split decomposition and consequently have a unique split decomposition tree [24].

In [30] and [31], a polynomial time algorithm to obtain an optimal proper coloring of a graph with a given split decomposition tree was proposed. To explain it, we need some technical definitions.

Let $G$ be a graph and let $w : V(G) \to \mathbb{N}$ be a weight function. A $w$-weighted coloring is an assignement, for each vertex $v$, of a set $C_v$ with $w(v)$ colors such that, for every edge $xy$, $C_x \cap C_y = \emptyset$. Let $C_{V'}$ be the set of colors used in a subset $V' \subseteq V(G)$.

Let $D(G, w, V')$ be the set of all pairs $(a, b) \in \mathbb{N}^2$ such that there is a $w$-weighted coloring $C$ of $G$ with $a + b$ colors and $|C_{V'}| = a$. Let $T$ be a split decomposition tree of $G$ and let $r$ be its root. Given a node $h \neq r$ of $T$, let $D(h) = D(G_h - v_h, w, N_{G_h}(v_h))$ be the $D$-set of $h$, where, for all $v \in V_h \setminus \{v_h\}$, $w(v) = 1$.

With these definitions, we can summarize the algorithm of [30] and [31].

For each node $h$ of $T$, they obtain the set $D(h)$ of all pairs $(a, b)$ such that there is a proper coloring of $G_h - v_h$ with $a + b$ colors where the neighbors of $v_h$ in $G_h$ receive $a$ colors. This is done by the following procedure.

Let $b \in \{0, \ldots, n\}$ be the number of colors we wish to reserve for $v_h$ (that is, these colors cannot appear in the neighborhood of $v_h$ in $G_h$). Let $c \in \{1, \ldots, 2n\}$ be an upper bound to the number of colors in $G_h$. For each child $i$ of $h$, let $a_i$ be the smallest integer such that there exists a coloring of $G_i - v_i$ with at most $c$ colors where the neighbors of $v_i$ in $G_i$ use $a_i$ colors. We can compute $a_i$ with the set $D(i)$. Let $w$ be the following weight function for $G^*_h$: $w(v_h) = b$, $w(v_i) = a_i$ for every child $i$ of $h$ and $w(x) = 1$ for the remaining vertices of $G^*_h$. Compute an optimal $w$-weighted coloring of $G^*_h$ and let $m_c$ be the minimum between the number of colors and the value of $c$. That is, for every value of $c$, we have a value $m_c$. Let $m_b$ be the minimum over all values $m_c$. We then add the pair $(m_b - b, b)$ in $D_h$. After all possible values for $b$, we have the set $D(h)$. When $h$ is the root $r$, then $b$ can assume only one value $b = 0$, since $G_r$ has no parent marker.

How we can obtain a proper coloring of $G_h$ given a $w$-weighted coloring of $G^*_h$? The vertices with weight 1 receive the same color, the neighbors of $v_i$ in $G_i$ receive the colors of $v_i$ for every child $i$ of $h$ and the remaining vertices can receive the remaining colors.

For more details, see [30] and [31]. Observe that, if the number of vertices in $G^*_h$ is bounded by a constant for every node $h$ of $T$, then this procedure has polynomial time complexity.

In this paper, we change this algorithm a little bit to obtain an optimal acyclic coloring of $G$, proving Theorem 1.3.

The main idea behind our modification is that, given a graph $G$ decomposable into $G_1$ and $G_2$, if $G$ has a bicolored cycle with vertices of $G_1$ and $G_2$, then $G_1$ or $G_2$ also has a bicolored cycle. Then our procedure follows the main steps of the algorithm above, but computing $w$-weighted acyclic colorings, instead of $w$-weighted proper colorings.

## 6 Some technical proofs

We now provide some proofs of the most important results of the paper. Firstly, we need to state a definition and recall a theorem from [6].

**Definition 6.1.** *Let $G = (V, E)$ be a graph. A subset $M$ of $V$ with $1 \leq |M| \leq |V|$ is called a* module *if each vertex in $V - M$ is either adjacent to all vertices of $M$ or to none of them. A module $M$ is called a* homogeneous set *if $1 < |M| < |V|$. The graph obtained from $G$ by* *shrinking every maximal homogeneous set to one single vertex is called the* characteristic graph *of* $G$.

A graph is called *split graph* if its vertex set has a partition $(K, S)$ such that $K$ induces a clique and $S$ induces an independent set.

**Lemma 6.2** ([6]). *A p-connected graph $G$ is separable if and only if its characteristic graph is a split graph.*

The rest of the paper is dedicated to prove Theorem 4.3 and lemmas from Sections 2 and 3.

### 6.1 Nonrepetitive colorings

We start with the proofs of Lemmas 3.2, 3.3 and 3.4.

*Proof of Lemma 3.2.* If $G = G_1 \cup G_2$, then every color of $G_1$ can be used in $G_2$, and vice-versa. Thus, $\pi(G) = \max\{\pi(G_1), \pi(G_2)\}$. So, let $G = G_1 \vee G_2$. Suppose that $|V(G_1)| \geq 2$ and $|V(G)| \geq 2$. Let $a_1, b_1 \in V(G_1)$ and $a_2, b_2 \in V(G_2)$. Suppose that $a_1$ and $b_1$ receive color $C_1$ and that $a_2$ and $b_2$ receive color $C_2$. Then we have the bicolored $P_4$ $a_1a_2b_1b_2$, which is the repetition pattern $C_1C_2C_1C_2$; a contradiction. So, (a) all vertices of $G_1$ have distinct colors; or (b) all vertices of $G_2$ have distinct colors. □

*Proof of Lemma 3.3.* Let $G$ be a spider with partition $(R, C, S)$, such that $|C| = |S| = k$. A minimum acyclic coloring of $G$ can be easily obtained from an acyclic coloring of $G[R]$, by assigning a new color for each vertex in $C$ and finally by coloring each vertex of $S$ with any appropriated avaiable color of $C$. Thus, $\chi_a(G) = \chi_a(G[R]) + k$.

On the other hand, to produce a star coloring of $G$, we first color optimally $G[R]$ and then assign one new color to each vertex of $C$. If $G$ is thin, we color each vertex of $S$ with any appropriated available color of $C$. If $G$ is thick and $R \neq \emptyset$, then we use one of the colors of $R$ to color every vertex of $S$. Then, $\chi_{st}(G) = \chi_{st}(R) + k$. If $G$ is thick and $R = \emptyset$, then we have to add a new color and assign it to every vertex of $S$. By consequence, $\chi_{st}(G) = k + 1$. The same arguments can be used to $\pi(G)$. □

### 6.2 Clique coloring

*Proof of Lemma 3.5.* The proof is direct if $G = G_1 \cup G_2$. If $G$ is the join of two graphs $G_1$ and $G_2$, then it is easy to see that every maximal clique of $G$ must have vertices of $G_1$ and $G_2$, since, for every clique $C$ of $G_1$, $C \cup \{v_2\}$ (where $v_2 \in G_2$) is a clique of $G$. Then, coloring the vertices of $G_1$ with color 1 and the vertices of $G_2$ with color 2, we have that every maximal clique receives two colors.

Now suppose that $G$ is a quasi-spider with partition $(R, C, S)$. Suppose first that $R$ is not empty. The same argument below shows that there is no maximal clique of $G$ with vertices of $R$ and no vertex of $C$ and that there is no maximal clique of $G$ with vertices of $C$ and no vertex of $R$ or no vertex of $S$. Since there is no clique with two vertices of $S$, we can obtain a clique coloring of $G$ by coloring the vertices of $R$ and $S$ with color 1 and the vertices of $C$ with color 2.

Suppose now that $R$ is empty. In this case, is is possible that $C$ is a maximal clique. Let $H = K_2$ or $H = \overline{K_2}$ be the subgraph that replaced a vertex of $C \cup S$ in the definition of quasi-spider. Let $x \in C - H$ be a vertex of $C$ that is not in $H$. Let $N(x)$ be the set of neighbors of $x$ in $S$. It is easy to see that coloring $C - \{x\}$ and $N(x)$ with color 1 and $x$ and $S - N(x)$ with color 2, we have that every maximal clique receives two colors. Then we have a 2-clique-coloring of $G$. □

*Proof of Lemma 4.2.* At first, suppose that $G - H$ is not empty. Then $H$ is a separable p-component and, by Lemma 6.2, the characteristic graph of $H$ is a split graph ($H_1$ "*reduces*" to a clique and $H_2$ "*reduces*" to an independente set). Also remember that every vertex of $H_2$ has a neighbor in $H_1$. Hence, if two vertices of $H_2$ induces an edge, then they are in the same homogeneous set and then they have a common neighbor in $H_1$. Consequently, there is no maximal clique with vertex set contained in $H_2$.

It is easy to see that there is no maximal clique of $G$ with vertices of $G - H$ and no vertex of $H_1$, since, for every clique $C$ of $G - H$, $C \cup \{v\}$ (where $v \in H_1$) is a clique of $G$. The same argument shows that there is no maximal clique of $G$ with vertices of $H_1$ and no vertex of $H_2$ or no vertex of $G - H$.Then, we can obtain a clique coloring of $G$ by coloring the vertices of $G - H$ and $H_2$ with color 1 and the vertices of $H_1$ with color 2.

Now, suppose that $G - H$ is empty. Since $H$ is a p-connected $(q, q-4)$-graph, then $H$ has at most $q - 1$ vertices. Since $q$ is fixed, we can generate all possible clique-colorings in constant time and obtain the clique chromatic number. □

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