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ON CIRCULAR-ARC GRAPHS HAVING A MODEL
WITH NO THREE ARCS COVERING THE CIRCLE

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ABSTRACT

An *interval graph* is the intersection graph of a finite set of intervals on a line and a *circular-arc graph* is the intersection graph of a finite set of arcs on a circle. While a forbidden induced subgraph characterization of interval graphs was found fifty years ago, finding an analogous characterization for circular-arc graphs is a long-standing open problem. In this work, we study the intersection graphs of finite sets of arcs on a circle no three of which cover the circle, known as *normal Helly circular-arc graphs*. Those circular-arc graphs which are minimal forbidden induced subgraphs for the class of normal Helly circular-arc graphs were identified by Lin, Soulignac, and Szwarcfiter, who also posed the problem of determining the remaining minimal forbidden induced subgraphs. In this work, we solve their problem, obtaining the complete list of minimal forbidden induced subgraphs for the class of normal Helly circular-arc graphs.

KEYWORDS. Forbidden subgraphs, interval graphs, normal Helly circular-arc graphs.

Main area. Theory and Algorithms in Graphs

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1. Introduction

The intersection graph of a finite family of sets has one vertex representing each member of the family, two vertices being adjacent if and only if the members of the family they represent have nonempty intersection. An *interval graph* is the intersection graph of a finite set of intervals on a line. Fifty years ago, Lekkerkerker and Boland (1962) found their celebrated characterization of interval graphs by minimal forbidden induced subgraphs. An interesting special case of interval graphs are the *unit interval graphs*, which are the intersection graphs of finite sets of all closed (or all open) intervals having the same length, and for which there is a forbidden induced subgraph characterization due to Roberts (1969) (see also Frankl and Maehara, 1987).

A *circular-arc graph* is the intersection graph of a finite set of arcs on a circle. Despite their similarity in definition to interval graphs, characterizing circular-arc graphs by forbidden induced subgraphs is a long-standing open problem (see Hadwiger and Debrunner, 1964, p. 54; Klee, 1969). Tucker (1971) pioneered the study of circular-arc graphs and some important subclasses, like *unit circular-arc graphs* (defined analogously to unit interval graphs) and *proper circular-arc graphs*, which are those intersection graphs of finite sets of arcs on a circle such that none of the arcs is contained in another of the arcs. Moreover, Tucker (1974) found the minimal forbidden induced subgraph characterizations of both unit circular-arc graphs and proper circular-arc graphs. Since then, the problem of characterizing circular-arc graphs and some of its subclasses by forbidden induced subgraphs or some other kinds of obstructions has attracted considerable attention (Bang-Jensen and Hell, 1994; Bonomo et al., 2009; Feder et al., 1999; Hell and Huang, 2004; Joeris et al., 2011; Lin et al., 2007, 2011; Trotter and Moore, 1976).

We say that a set of arcs on a circle *covers the circle* if the arcs of the set collectively cover every point of the circle. It is easy to see that every circular-arc graph is the intersection graph of a finite set of arcs no single arc of which covers the circle. Intersection graphs of finite sets of arcs on a circle no two arcs of which cover the circle are known as *normal circular-arc graphs*. The class of normal circular-arc graphs properly contains the class of proper circular-arc graphs as shown by Tucker (1974) and was studied in the context of co-bipartite graphs by Hell and Huang (2004). Some partial characterizations by minimal forbidden induced subgraphs for the class of normal circular-arc graphs are known (see Bonomo et al., 2009, especially Section 5).

In this work, we study the intersection graphs of finite sets of arcs on a circle no three arcs of which cover the circle, known as *normal Helly circular-arc graphs* (Lin et al., 2007). Notice that, for any set of arcs on a circle having at least three arcs, the property of not having three arcs covering the circle precludes also the existence of fewer than three arcs covering the circle. If $\mathscr{A}$ is a set of arcs on a circle, then: (i) $\mathscr{A}$ is said *normal* if it has no two arcs covering the circle, (ii) $\mathscr{A}$ is said *Helly* if every nonempty subset of $\mathscr{A}$ consisting of pairwise intersecting arcs has nonempty total intersection, and (iii) $\mathscr{A}$ is said *normal Helly* if $\mathscr{A}$ is both normal and Helly. In turns out that normal Helly circular-arc graphs can be defined as the intersection graphs of finite normal Helly sets of arcs on a circle. Indeed, it follows from Theorem 1 of Lin and Szwarcfiter (2006) that this definition of normal Helly circular-arc graphs is equivalent to the one we use along this work (i.e., the intersection graphs of finite sets of arcs on a circle no three arcs of which cover the circle).

Some previous works related to normal Helly circular-arc graphs are the following. Tucker (1975) gave an algorithm that outputs a proper coloring of any given normal Helly circular-arc graph using at most $3\omega/2$ colors, where ω denotes the maximum size of a set of pairwise adjacent vertices. In Lin et al. (2010), normal Helly circular-arc graphs arose naturally when studying convergence of circular-arc graphs under the clique operator. The *boxicity* of a graph G is the minimum k such that G is the intersection graph of a family of k-dimensional boxes (i.e., of Cartesian products of k closed intervals); it was shown by Bhowmick and Sunil Chandran (2011) that normal Helly circular-arc graphs have boxicity at most 3.

Recently, Lin, Soulignac, and Szwarcfiter (2011) undertook a thorough study of normal Helly circular-arc graphs, drawing many parallels between these graphs and interval graphs. In that

work, they determined all those circular-arc graphs which are minimal forbidden induced subgraphs for the class of normal Helly circular-arc graphs and posed the problem of finding the remaining minimal forbidden subgraphs for the class of normal Helly circular-arc graphs (i.e., those which are not circular-arc graphs). In this work, we solve their problem, providing the complete list of minimal forbidden induced subgraphs for the class of normal Helly circular-arc graphs.

2. Preliminaries

All graphs in this work are finite, undirected, and have no loops and no parallel edges. The vertex and edge sets of a graph G will be denoted by $V(G)$ and $E(G)$, respectively. We denote by $\overline{G}$ the *complement* of G, by $N_G(v)$ the *neighborhood* of a vertex v in G, and by $N_G[v]$ its *closed neighborhood* $N_G(v) \cup \{v\}$. Vertex v is *isolated* if $N_G(v) = \emptyset$ and *universal* if $N_G[v] = V(G)$. The subgraph *induced* by a set of vertices S is denoted by $G[S]$. If H is an induced subgraph of G, we say that G *contains an induced* H. Paths in this work are meant to have at least one vertex. An *a,b-path* is a path whose *endpoints* are a and b; the remaining vertices of the path are the *interior vertices*. A *chord* of a path or cycle Z is any edge not in Z joining two vertices of Z. A *chordless path* is a path having no chords and a *chordless cycle* is a cycle on four or more vertices having no chords. We denote by P_n (resp. C_n) the chordless path (resp. cycle) on n vertices. A graph is *chordal* if it has no chordless cycle. A *clique* is a set of pairwise adjacent vertices. Two vertices are in the same *component* of a graph G if there is a path joining them in G. If S is a set, we denote its cardinality by $|S|$. For standard notation and terminology not defined here, we refer to West (2001).

Let $\mathscr{G}$ be a graph class. A graph H is a *forbidden induced subgraph* of $\mathscr{G}$ if no member of $\mathscr{G}$ contains an induced H. A class $\mathscr{G}$ of graphs is *hereditary* if every induced subgraph of every member of $\mathscr{G}$ is also a member of $\mathscr{G}$. If H is a forbidden induced subgraph of a hereditary graph class $\mathscr{G}$, then H is a *minimal forbidden induced subgraph* of $\mathscr{G}$ if every induced subgraph of H different from H is a member of $\mathscr{G}$. Clearly, a hereditary graph class is completely determined by the its minimal forbidden induced subgraphs: the graph class consists exactly of those graphs containing no induced minimal forbidden induced subgraph. The following celebrated result gives the complete list of minimal forbidden induced subgraphs for the class of interval graphs.

Theorem 1 (Lekkerkerker and Boland, 1962). *The minimal forbidden induced subgraphs for the class of interval graphs are: bipartite claw, umbrella, k-net for every $k \geq 2$, k-tent for every $k \geq 3$, and C_k for every $k \geq 4$ (see Figure 1).*
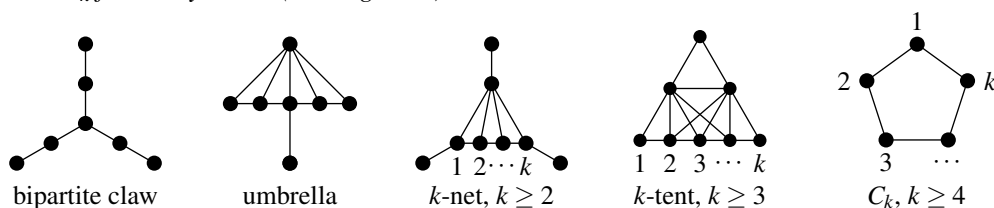

Fig. 1: All minimal forbidden induced subgraphs for the class of interval graphs

In contrast with the situation for interval graphs, the problem of characterizing circular-arc graphs by forbidden induced subgraphs is still open. Some minimal forbidden induced subgraphs for the class of circular-arc graphs are G_1, G_2, G_3, G_4, domino, $\overline{G_6}$, and C_k^* for every $k \geq 4$, where C_k^* denotes the graph that arises from C_k by adding an isolated vertex (see Figure 2); it follows, for instance, from our main result (Theorem 6) that these graphs are also minimal forbidden induced subgraphs for the class of normal Helly circular-arc graphs. For each $k \geq 4$, the graph *k-wheel*, that arises from C_k by adding a universal vertex (see Figure 2), is a circular-arc graph but also a minimal forbidden induced subgraph for the class of normal Helly circular-arc graphs. In what follows, we use *net* and *tent* as shorthands for 2-net and 3-tent, respectively.

If a graph G is the intersection graph of a finite family $\mathscr{F}$, then $\mathscr{F}$ is called an *intersection model* of $\mathscr{F}$. Moreover, if $\mathscr{F}$ consists of intervals on a line, then $\mathscr{F}$ is called an *interval model* of G, whereas if $\mathscr{F}$ consists of arcs on a circle, then $\mathscr{F}$ is called a *circular-arc model* of G. Let G

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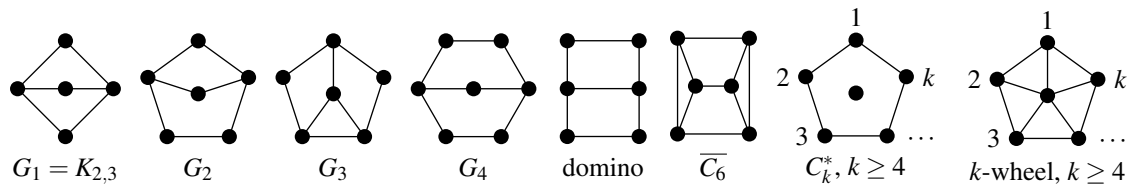
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Fig. 2: Some minimal forbidden induced subgraphs for the class of normal Helly circular-arc graphs

be an interval graph containing sets of vertices A and B. The pair A, B is said *left-right* if G has an interval model where all intervals corresponding to vertices of A have the same left endpoint and no other endpoints are further to the left and all vertices of B have the same right endpoint and no other endpoints are further to the right (de Figueiredo et al., 1997). A vertex v of G is an *end* of G if the pair $\{v\}, \emptyset$ is left-right. Gimbel (1988) gave the following characterization of end vertices.

Theorem 2 (Gimbel, 1988). *Let G be an interval graph. If v is a vertex of G, then v is an end vertex of G if and only if G contains none of the graphs in Figure 3 as an induced subgraph where the filled vertex represents v.*
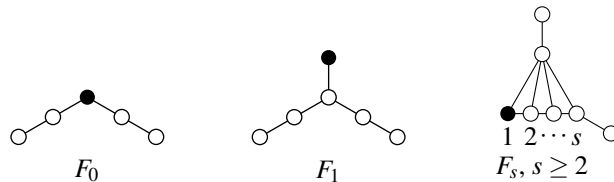

Fig. 3: Forbidden induced subgraphs characterizing end vertices. Filled vertices are not end vertices

Left-right pairs were characterized by de Figueiredo et al. (1997), as follows.

Theorem 3 (de Figueiredo et al., 1997). *Let G be an interval graph and $A, B \subseteq V(G)$. Then, A, B is left-right if and only if each of the following assertions holds:*

  (i) A and B both are cliques.

  (ii) Each vertex of $A \cup B$ is an end vertex.

 (iii) Given any pair of vertices u and v, both in A or both in B, there is no any chordless path on four vertices in G, with u and v as internal vertices of the path;

 (iv) For each $a \in A$ and $b \in B$, there is no chordless a,b-path in G, together with a vertex v which is adjacent in G with no vertex on the path.

In the next section, we will make use of the theorem below, whose proof is an easy adaptation of the proof of Theorem 6 in Bonomo et al. (2009). We give the adapted proof for completeness.

Theorem 4 (adapted from Theorem 6 of Bonomo et al., 2009). *Let G be a graph containing no induced G_1, G_2, G_3, G_4, domino, k-wheel for any $k \geq 4$, or C_k^* for any $k \geq 4$. If C is a chordless cycle of G and $v \in V(G) \setminus V(C)$, then the neighbors of v in $V(C)$ induce a chordless path in G.*

Proof. First, notice that $N_G(v) \cap V(C) \neq \emptyset$ and $N_G(v) \cap V(C) \neq V(C)$ because G contains no induced k-wheel and no induced C_k^* for any $k \geq 4$. Suppose, by the way of contradiction, that the neighbors of v in $V(C)$ do not induce a chordless path in G. Then, the nonneighbors of v in $V(C)$ do not induce a connected subgraph in G and let P^1 and P^2 be two components of the subgraph of G induced by the nonneighbors of v in $V(C)$. By construction, P^1 and P^2 are chordless paths and, by symmetry, assume that $|V(P^1)| \geq |V(P^2)|$. Let x_1 and x_2 (resp. y_1 and y_2) be the neighbors of the endpoints of P^1 (resp. P^2) in $V(C)$. Without loss of generality, assume that x_1, y_1, x_2, y_2 are labeled in such a way that, in the subgraph of G induced by $V(C) \setminus (V(P^1) \cup V(P^2))$, x_1 and y_2 are in the same component and also x_2 and y_1 are in the same component. Let $k = |V(P^2)| + 3$; clearly, $k \geq 4$.

Suppose first that $x_1 = y_2$ and $x_2 \neq y_1$. On the one hand, if $|V(P^1)| = 1$, then also $|V(P^2)| = 1$ and $V(P^1) \cup V(P^2) \cup \{x_1, x_2, y_1, v\}$ would induce domino or G_3, depending on whether $|V(C)| \geq 6$ or not, respectively, a contradiction. On the other hand, if $|V(P^1)| \geq 2$, then $V(P^2) \cup \{y_1, y_2, v, x\}$ would induce C_k^* for any vertex x of P^1 nonadjacent to x_1, a contradiction. These contradictions show, by symmetry, that either $x_1 = y_2$ and $x_2 = y_1$, or $x_1 \neq y_2$ and $x_2 \neq y_1$. Notice that we can

assume that $|V(P^1)| \leq 2$ because otherwise $V(P^2) \cup \{y_1, y_2, v, x\}$ would induce C_k^* in G for any vertex x in P^1 nonadjacent to both x_1 and x_2. Therefore, if $x_1 = y_2$ and $x_2 = y_1$, then $V(C) \cup \{v\}$ would induce G_1, or G_2, or G_4 in G, a contradiction. Finally, if $x_1 \neq y_2$ and $x_2 \neq y_1$, then $V(P^2) \cup \{y_1, y_2, v, x\}$ would induce C_k^* for any vertex x in P^1, a contradiction. These contradictions arose from assuming that the neighbors of v in $V(C)$ do not induce a chordless path in $V(C)$. ∎

3. Forbidden induced subgraph characterization of normal Helly circular-arc graphs

The main result of this section is Theorem 6 which characterizes normal Helly circular-arc graphs by minimal forbidden induced subgraphs. Lin, Soulignac, and Szwarcfiter (2011) solved this problem partially, by restricting themselves to circular-arc graphs; i.e., they found the list of all those circular-arc graphs which are minimal forbidden induced subgraphs for the class of normal Helly circular-arc graphs, as follows.

Theorem 5 (Lin et al., 2011). *Let H be a circular-arc graph. If H is a minimal forbidden induced subgraph for the class of normal Helly circular-arc graphs, then H is isomorphic to one of the following graphs: bipartite claw, umbrella, net, k-tent for some $k \geq 3$, or k-wheel some $k \geq 4$.*

The remaining of this section is devoted the state and prove our main result below.

Theorem 6. *A graph G is a normal Helly circular-arc graph if and only if G contains no induced bipartite claw, umbrella, k-net for any $k \geq 2$, k-tent for any $k \geq 3$, k-wheel for any $k \geq 4$, G_1, G_2, G_3, G_4, domino, $\overline{C_6}$, or C_k^* for any $k \geq 4$ (see Figures 1 and 2).*

Proof. The necessity is clear. So, assume that G contains no induced bipartite claw, umbrella, k-net for any $k \geq 2$, k-tent for any $k \geq 3$, k-wheel for any $k \geq 4$, G_1, G_2, G_3, G_4, domino, $\overline{C_6}$, or C_k^* for any $k \geq 4$. Because of Theorem 5, in order to prove that G is a normal Helly circular-arc graph, it suffices to show that G is a circular-arc graph. If G is chordal, then G is an interval graph by Theorem 1 and, in particular, a circular-arc graph. Thus, we assume, without loss of generality, that G has some chordless cycle $C = v_1 v_2 \ldots v_n v_1$ for some $n \geq 4$. In what follows, vertex and set subindices should be understood modulo n.

For each $i \in \{1, \ldots, n\}$, we define the sets A_i, B_i, O_i, and T_i as follows:
- A vertex $v \in V(G)$ belongs to A_i if and only if v is adjacent to v_{i-1} and v_i and also adjacent to some vertex w which is adjacent to v_i and nonadjacent to v_{i-1}. (For instance, w may be v_{i+1}.)
- A vertex $v \in V(G)$ belongs to B_i if and only if v is adjacent to v_i and v_{i+1} an also adjacent to some vertex w which is adjacent to v_i and nonadjacent to v_{i+1}. (For instance, w may be v_{i-1}.)
- A vertex $v \in V(G)$ belongs to O_i if and only if $N_G(v) \cap V(C) = \{v_i\}$.
- A vertex $v \in V(G)$ belongs to T_i if and only if $N_G(v) \cap V(C) = \{v_i, v_{i+1}\}$ and $v \notin B_i \cup A_{i+1}$.

Claims 1 to 17 below will be used to build a circular-arc model for G; see paragraph 'Constructing a circular-arc model for G' immediately after the proof of Claim 17.

Claim 1. $V(G) \setminus V(C) = \bigcup_{i=1}^{n}(A_i \cup B_i \cup O_i \cup T_i)$.

Proof. Since C is a chordless cycle, it is clear by definition that no vertex in $\bigcup_{i=1}^{n}(A_i \cup B_i \cup O_i \cup T_i)$ can belong to $V(C)$. Conversely, let $v \in V(G) \setminus V(C)$. Since G contains no induced C_n^*, v has at least one neighbor in $V(C)$. If $|N_G(v) \cap V(C)| = 1$, then $N_G(v) \cap V(C) = \{v_i\}$ for some $i \in \{1, 2, \ldots, n\}$ and, by definition, $v \in O_i$. If $|N_G(v) \cap V(C)| = 2$, then, by Theorem 4, $N_G(v) \cap V(C) = \{v_i, v_{i+1}\}$ for some $i \in \{1, 2, \ldots, n\}$ and, by definition, $v \in T_i \cup B_i \cup A_{i+1}$. Finally, if $|N_G(v) \cap V(C)| \geq 3$, then, by Theorem 4, v is adjacent to v_{i-1}, v_i, and v_{i+1} for some $i \in \{1, 2, \ldots, n\}$ and, by definition, $v \in A_i$. □

In each of the claims below, i is any integer belonging to $\{1, 2, \ldots, n\}$.

Claim 2. *Every vertex v of G which is adjacent simultaneously to v_{i-1}, v_i, and v_{i+1}, is also adjacent to every vertex o in O_i.*

Proof. As v is adjacent to v_{i-1}, v_i, and v_{i+1}, Theorem 4 implies that $N_G(v) \cap V(C)$ induces a chordless path $P = v_p v_{p+1} \ldots v_{p+m}$ for some $p \in \{1, \ldots, n\}$ and some $m \leq n-2$ such that v_i is an interior vertex of P. Thus, if v were nonadjacent to some $o \in O_i$, then $\{v, v_{p+m}, v_{p+m+1}, \ldots, v_p, o\}$ would induce C_{n-m+2}^* in G, where $n-m+2 \geq 4$, a contradiction. This contradiction proves the claim. □

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Claim 3. $G[A_i \cup O_i \cup B_i]$ and $G[B_i \cup T_i \cup A_{i+1}]$ are interval graphs.

Proof. By Theorem 1, it suffices to prove that $G[A_i \cup O_i \cup B_i]$ and $G[B_i \cup T_i \cup A_{i+1}]$ contain no induced C_k for any $k \geq 4$. If there were any chordless cycle C' in $G[A_i \cup O_i \cup B_i]$ or $G[B_i \cup T_i \cup A_{i+1}]$, then, since every vertex in $A_i \cup O_i \cup B_i \cup T_i \cup A_{i+1}$ is adjacent to v_i, $V(C') \cup \{v_i\}$ would induce k-wheel in G where $k = |V(C')| \geq 4$, a contradiction. This contradiction proves the claim. □

Claims 4 to 7 below, together with Theorem 3 and Claim 3, will prove that A_i, B_i is a left-right pair of $G[A_i \cup O_i \cup B_i]$ (Claim 8).

Claim 4. A_i and B_i are cliques of G.

Proof. Let $a_1, a_2 \in A_i$ such that $a_1 \neq a_2$ and assume, by the way of contradiction, that a_1 and a_2 are nonadjacent. By definition, for each $j \in \{1, 2\}$, there is some neighbor w_j of a_j such that w_j is adjacent to v_i and nonadjacent to v_{i-1}. Notice that a_2 is nonadjacent to w_1, since otherwise $\{v_i, v_{i-1}, a_1, w_1, a_2\}$ would induce 4-wheel in G. Symmetrically, a_1 is nonadjacent to w_2 and, necessarily, $w_1 \neq w_2$. Moreover, w_1 is nonadjacent to w_2, since otherwise $\{v_i, v_{i-1}, a_1, w_1, w_2, a_2\}$ would induce 5-wheel in G. We notice that a_1 and a_2 cannot be simultaneously adjacent to v_{i-2}, since otherwise $\{v_{i-1}, v_i, a_1, v_{i-2}, a_2\}$ would induce 4-wheel in G. Without loss of generality, assume that a_2 is nonadjacent to v_{i-2}. If a_1 were adjacent to v_{i-2}, then either $\{v_i, v_{i-1}, v_{i-2}, w_1, a_1\}$ would induce 4-wheel in G or $\{v_i, v_{i-1}, a_1, a_2, v_{i-2}, w_1\}$ would induce tent in G, depending on whether w_1 is adjacent to v_{i-1} or not, respectively. Therefore, also a_1 is nonadjacent to v_{i-2}. Notice that at least one of w_1 and w_2 is adjacent to v_{i-2}, since otherwise $\{v_{i-2}, w_1, a_1, v_{i-1}, a_2, w_2, v_i\}$ would induce umbrella in G. If both w_1 and w_2 are adjacent to v_{i-2}, then $\{v_i, v_{i-1}, v_{i-2}, w_1, w_2\}$ induces G_1 in G, whereas if only w_1 is adjacent to v_{i-2}, then $\{w_1, v_{i-2}, v_{i-1}, a_1, w_2\}$ induces C_4^* in G. These contradictions arose from assuming that a_1 and a_2 were nonadjacent. We conclude that A_i is a clique of G and, by symmetry, that also B_i is a clique of G. □

Claim 5. There are no two vertices u and v, both in A_i or both in B_i, such that u and v are the internal vertices of a chordless path on four vertices in $G[A_i \cup O_i \cup B_i]$.

Proof. Suppose, by the way of contradiction, that the claim is false; i.e., there are two vertices u and v, both in A_i or both in B_i, such that u and v are the internal vertices of a chordless path P on four vertices in $G[A_i \cup O_i \cup B_i]$. By symmetry, we assume, without loss of generality, that u and v both belong to A_i and let $a_1 = u$ and $a_2 = v$. As we proved in Claim 4 that A_i and B_i are cliques, the endpoints of P either both belong to O_i or one belongs to O_i and the other one to B_i. Let us assume, without loss of generality, that $P = xa_1a_2o$ where $o \in O_i$ and either $x \in O_i$ or $x \in B_i$.

We assert that x is adjacent to v_i and nonadjacent to both v_{i-1} and v_{i-2}. The assertion is clearly true if $x \in O_i$; so, in what remains of this paragraph, we assume, without loss of generality, that $x \in B_i$. By definition, x is adjacent to v_i and v_{i+1} and, by construction, nonadjacent to o. Hence, Claim 2 implies x is nonadjacent to v_{i-1}. Moreover, x is nonadjacent to v_{i-2}, since otherwise either $\{a_1, v_{i-2}, v_{i-1}, v_i, x\}$ would induce 4-wheel in G or $\{v_{i-2}, v_{i-1}, a_1, x, o\}$ would induce C_4^* in G, depending on whether a_1 is adjacent to v_{i-2} or not, respectively. This proves the assertion.

If none of a_1 and a_2 were adjacent to v_{i-2}, then $\{v_{i-2}, v_{i-1}, x, a_1, a_2, o\}$ would induce net in G. If both a_1 and a_2 were adjacent to v_{i-2}, then $\{v_{i-2}, v_i, x, a_1, a_2, o\}$ would induce tent in G. Finally, if exactly one of a_1 and a_2 were adjacent to v_{i-2}, then $\{v_i, a_1, v_{i-1}, a_2, x, o, v_{i-2}\}$ would induce 4-tent in G. These contradictions prove the claim. □

Claim 6. For each $a \in A_i$ and each $b \in B_i$, there is no chordless a, b-path in $G[A_i \cup O_i \cup B_i]$ together with a vertex $v \in A_i \cup O_i \cup B_i$ such that v is adjacent to no vertex of the path.

Proof. Suppose, by the way of contradiction, that the claim is false and let $P = x_0 x_1 \ldots x_p$ be a chordless path in $G[A_i \cup O_i \cup B_i]$ with minimum number of vertices such that $x_0 \in A_i$, $x_p \in B_i$, and there is a vertex $v \in A_i \cup O_i \cup B_i$ such that v is adjacent to no vertex of P. As A_i and B_i are cliques (by Claim 4), $V(P) \cap A_i \subseteq \{x_0, x_1\}$ and $V(P) \cap B_i \subseteq \{x_{p-1}, x_p\}$. Moreover, if $x_1 \in A_i$, then $P' = x_1 x_2 \ldots x_p$ would be a chordless path in $G[A_i \cup O_i \cup B_i]$ having less vertices than P and such that $x_1 \in A_i$, $x_p \in B_i$, and v would be adjacent to no vertex of P', which would contradict

the choice of P. Hence, x_0 is the only vertex of A_i in P and, symmetrically, x_p is the only vertex of B_i in P; i.e., $x_1, x_2, \ldots, x_{p-1} \in O_i$. Since A_i and B_i are cliques and since v is nonadjacent to $x_0 \in A_i$ and $x_p \in B_i$, $v \in O_i$ holds. Let $a = x_1$, $b = x_p$, and $o = v$. Since $a \in A_i$ and $b \in B_i$ are nonadjacent to o, Claim 2 implies that a is nonadjacent to v_{i+1} and b is nonadjacent to v_{i-1}. In particular, $a \neq b$ and $p \geq 1$. Theorem 4 guarantees that there are some $q, r \in \{1, \ldots, n\}$ such that $N_G(a) \cap V(C) = \{v_i, v_{i-1}, \ldots, v_q\}$ and $N_G(b) \cap V(C) = \{v_i, v_{i+1}, \ldots, v_r\}$. If a and b had no common neighbor in $V(C)$ apart from v_i, then $V(P) \cup \{v_r, v_{r+1}, \ldots, v_q, o\}$ would induce C_k^* in G for some $k \geq 4$, a contradiction. Hence, a and b have some common neighbor in $V(C)$ different from v_i and necessarily $v_r \in N_G(a)$. Notice that $v_r \neq v_i$ because b is adjacent to v_{i+1} and that $v_r \neq v_{i+1}$ because a is adjacent to v_r and nonadjacent to v_{i+1}. Therefore, if $p = 1$, then $V(P) \cup \{v_i, v_{i-1}, \ldots, v_r\}$ induces a k-wheel in G for some $k \geq 4$, a contradiction. On the contrary, if $p \geq 2$, then $V(P) \cup \{v_r, o\}$ induces C_{p+2}^* in G where $p + 2 \geq 4$, a contradiction. These contradictions prove the claim. $\square$

Claim 7. *Each vertex of $A_i \cup B_i$ is an end vertex in $G[A_i \cup O_i \cup B_i]$.*

Proof. By symmetry, it suffices to prove that each vertex of A_i is an end vertex of $G[A_i \cup O_i \cup B_i]$. Suppose, by the way of contradiction, that not every vertex of A_i is an end vertex of $G[A_i \cup O_i \cup B_i]$. By Theorem 2, there is some minimum nonnegative integer s such that $G[A_i \cup O_i \cup B_i]$ contains an induced F_s (see Figure 3) where the filled vertex is some vertex $a \in A_i$.

Assume first that $s = 0$; i.e., there is a chordless path $P = x_1 x_2 x_3 x_4 x_5$ in $G[A_i \cup O_i \cup B_i]$ where $x_3 = a$. Since Claim 4 ensures that A_i and B_i are cliques, either x_1 or x_5 belongs to O_i. By symmetry, assume that $x_1 \in O_i$. As x_4 and x_5 belong to the same component of $G[A_i \cup O_i \cup B_i] \setminus N_G[x_1]$ than $a \in A_i$, Claim 6 implies that none of x_4 and x_5 belongs to B_i, Necessarily, $x_5 \in O_i$ because x_5 is nonadjacent to $a \in A_i$ and A_i is a clique (by Claim 4). Symmetrically, from $x_5 \in O_i$ we deduce that $x_2 \notin B_i$. By Claim 5 applied to the chordless paths $x_1 x_2 a x_4$ and $x_2 a x_4 x_5$, it follows that none of x_2 and x_4 belongs to A_i. We conclude that $x_1, x_2, x_4, x_5 \in O_i$ and, consequently, either $\{v_{i-2}, v_i\} \cup V(P)$ induces umbrella or $\{v_{i-2}, v_{i-1}\} \cup V(P)$ induces bipartite claw, depending on whether a is adjacent to v_{i-1} or not, respectively. These contradictions prove that $s \neq 0$.

Assume now that $s = 1$; i.e., there is a chordless path $P = x_1 x_2 x_3 x_4 x_5$ in $G[A_i \cup O_i \cup B_i]$ such that the only neighbor of a in P is x_3. Since A_i and B_i are cliques, either x_1 or x_5 belongs to O_i. Without loss of generality, assume that $x_1 \in O_i$. Since x_3, x_4, and x_5 belong to the same component of $G[A_i \cup O_i \cup B_i] \setminus N_G[x_1]$ than $a \in A_i$, Claim 6 implies that none of x_3, x_4, and x_5 belongs to B_i. Notice that $x_3 \notin A_i$ (since otherwise s would be 0) and none of x_4 and x_5 belongs to A_i because both vertices are nonadjacent to a and A_i is a clique (by Claim 4). Necessarily, $x_3, x_4, x_5 \in O_i$. Symmetrically, from $x_5 \in O_i$, it follows that $x_1, x_2 \in O_i$. Thus, $V(P) \subseteq O_i$, which means that $\{v_{i-1}, a\} \cup V(P)$ induces a bipartite claw in G, a contradiction. This contradiction proves that $s \neq 1$.

It only remains to consider the case $s \geq 2$; i.e., there is a chordless path $P = a x_2 \ldots x_s x_{s+1}$ in $G[A_i \cup O_i \cup B_i]$ and two vertices $y_1, y_2 \in A_i \cup O_i \cup B_i$ whose neighborhoods in $A_i \cup O_i \cup B_i$ are $\{y_2\}$ and $\{a, x_2, \ldots, x_s\}$, respectively. As A_i and B_i are cliques, it holds that $y_1, x_{s+1} \in O_i \cup B_i$ and at least one of y_1 and x_{s+1} belongs to O_i. Since $y_1, y_2, x_2, x_3, \ldots, x_{s-1}$ belong to the same component of $G[A_i \cup O_i \cup B_i] \setminus N_G[x_{s+1}]$ than a and $x_2, x_3, \ldots, x_{s+1}$ belong to the same component of $G[A_i \cup O_i \cup B_i] \setminus N_G[y_1]$ than a, Claim 6 implies that the fact that at least one of y_1 and x_{s+1} belongs to O_i means that none of $x_2, x_3, \ldots, x_s, x_{s+1}, y_1$, and y_2 belongs to B_i. Hence, $x_2, y_2 \in A_i \cup O_i$ and, as A_i is a clique, $x_3, x_4, \ldots, x_{s+1}, y_1 \in O_i$. Notice that at least one of x_2 and y_2 belongs to A_i, since otherwise $\{v_{i-1}, a, x_2, x_3, \ldots, x_s, y_1, y_2\}$ would induce s-net in G.

Assume first that $x_2 \in A_i$ but $y_2 \notin A_i$. On the one hand, if $s \geq 3$, then $\{v_{i-1}, x_2, x_3, \ldots, x_{s+1}, y_1, y_2\}$ would induce $(s-1)$-net in G. On the other hand, if $s = 2$, then the following assertions hold: if a and x_2 were adjacent to v_{i-2}, then $\{v_{i-2}, v_i, a, x_2, x_3, y_1, y_2\}$ would induce 4-tent in G; if a and x_2 were nonadjacent to v_{i-2}, then $\{v_{i-2}, v_{i-1}, a, x_2, x_3, y_1, y_2\}$ would induce 3-net in G; if a were adjacent to v_{i-2} and x_2 were nonadjacent to v_{i-2}, then $\{v_{i-2}, a, x_2, x_3, y_1, y_2\}$ would induce net in G; if a were nonadjacent to v_{i-2} but x_2 were adjacent to v_{i-2}, then $\{v_{i-2}, v_{i-1}, v_i, a, x_2, x_3, y_1, y_2\}$ would induce 5-tent in G. These contradictions arose from assuming that $x_2 \in A_i$ but $y_2 \notin A_i$.

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Suppose now that $y_2 \in A_i$ but $x_2 \notin A_i$. If both vertices a and y_2 were nonadjacent to v_{i-2}, then $\{v_{i-2}, v_{i-1}, a, x_2, x_3, \ldots, x_{s+1}, y_1, y_2\}$ would induce $(s+1)$-net in G. If a and y_2 were adjacent to v_{i-2}, then $\{v_{i-2}, v_i, a, x_2, x_3, \ldots, x_{s+1}, y_1, y_2\}$ would induce $(s+2)$-tent in G. If a were adjacent to v_{i-2} but y_2 were nonadjacent to v_{i-2}, then $\{v_{i-2}, a, x_2, x_3, \ldots, x_{s+1}, y_1, y_2\}$ would induce s-net in G. If a were nonadjacent to v_{i-2} and y_2 were adjacent to v_{i-2}, then $\{v_{i-2}, v_{i-1}, v_i, a, x_2, x_3, \ldots, x_{s+1}, y_1, y_2\}$ would induce $(s+3)$-tent in G. These contradictions arose from assuming that $y_2 \in A_i$ but $x_2 \notin A_i$

Necessarily, both x_2 and y_2 belong to A_i. Notice that $s \geq 3$, since otherwise the chordless path $y_1 y_2 x_2 x_3$ in $G[A_i \cup O_i \cup B_i]$ would have both interior vertices in A_i, contradicting Claim 5. If none of x_2 and y_2 is adjacent to v_{i-2}, then $\{v_{i-2}, v_{i-1}, x_2, x_3, \ldots, x_{s+1}, y_1, y_2\}$ induces s-net in G. If both x_2 and y_2 are adjacent to v_{i-2}, then $\{v_{i-2}, v_i, x_2, x_3, \ldots, x_{s+1}, y_1, y_2\}$ induces $(s+1)$-tent in G. If, from x_2 and y_2, only y_2 is adjacent to v_{i-2}, then $\{v_{i-2}, v_{i-1}, v_i, x_2, x_3, \ldots, x_{s+1}, y_1, y_2\}$ induces $(s+2)$-tent in G. Finally, if from x_2 and y_2, only x_2 is adjacent to v_{i-2}, then $\{v_{i-2}, x_2, x_3, \ldots, x_{s+1}, y_1, y_2\}$ induces $(s-1)$-net in G. These contradictions arose from assuming that there was some vertex of A_i which was not an end vertex of $G[A_i \cup O_i \cup B_i]$. Thus, every vertex of A_i is an end vertex of $G[A_i \cup O_i \cup B_i]$ and, symmetrically, every vertex of B_i is also an end vertex of $G[A_i \cup O_i \cup B_i]$. □

Claim 8. A_i, B_i *is a left-right pair of* $G[A_i \cup O_i \cup B_i]$.

Proof. It follows from Theorem 3 and Claims 3 to 7. □

Claim 9. *If some vertex* $b \in B_i$ *is nonadjacent to some vertex* $x \in T_i \cup A_{i+1}$*, then* b *is nonadjacent to* v_{i+2} *and, for every neighbor* w *of* b *such that* w *is adjacent to* v_i *and nonadjacent to* v_{i+1}*, w *is also nonadjacent to* x*.*

Proof. Let b be any vertex in B_i being nonadjacent to some vertex $x \in T_i \cup A_{i+1}$. As $b \in B_i$, there is some neighbor w of b such that w is adjacent to v_i and nonadjacent to v_{i+1}. Necessarily, w is nonadjacent to x, since otherwise $\{v_i, v_{i+1}, b, w, x\}$ would induce 4-wheel in G. Suppose, by the way of contradiction, that b were adjacent to v_{i+2}. Therefore, by definition, $b \in A_{i+1}$. As A_{i+1} is a clique (by Claim 4) but x is nonadjacent to b, it holds that $x \notin A_{i+1}$. Consequently, x is nonadjacent to v_{i+2}. Besides, w is nonadjacent to v_{i+2}, since otherwise $\{b, v_i, v_{i+1}, v_{i+2}, w\}$ would induce a 4-wheel in G. We conclude that, if b were adjacent to v_{i+2}, then $\{w, v_i, v_{i+1}, v_{i+2}, b, x\}$ would induce tent in G, a contradiction. This contradiction proves that b is nonadjacent to v_{i+2}. As we already proved that w is nonadjacent to x, the proof of the claim is complete. □

Claims 10 to 12 below, together with Theorem 3 and Claims 3 and 4, will prove that B_i, A_{i+1} are left-right sets in $G[B_i \cup T_i \cup A_{i+1}]$ (Claim 13).

Claim 10. *There are no two vertices* u *and* v*, both in* B_i *or both in* A_{i+1}*, such that* u *and* v *are the internal vertices of a chordless path on four vertices in* $G[B_i \cup T_i \cup A_{i+1}]$*.*

Proof. Suppose, by the way of contradiction, that there are two vertices u and v, both in B_i or both in A_{i+1}, such that u and v are the internal vertices of a chordless path P on four vertices in $G[B_i \cup O_i \cup A_{i+1}]$. By symmetry, we assume, without loss of generality, that u and v both belong to B_i. As Claim 4 ensures that that B_i and A_{i+1} are cliques, the endpoints of P either both belong to T_i or one belongs to T_i and the other one to A_{i+1}. Assume, without loss of generality, that $P = x b_1 b_2 t$ where $b_1, b_2 \in B_i$, $t \in T_i$, and either $x \in A_i$ or $x \in T_i$. For each $j \in \{1, 2\}$, let w_j be a neighbor of b_j such that w_j is adjacent to v_i and nonadjacent to v_{i+1}. As $b_1 \in B_i$ is nonadjacent to $t \in T_i$ and $b_2 \in B_i$ is nonadjacent to $x \in T_i \cup A_{i+1}$, Claim 9 implies that b_1 and b_2 are nonadjacent to v_{i+2}, w_1 is nonadjacent to t, and w_2 is nonadjacent to x. If w_1 were adjacent to x, then, by definition of B_i, $x \in B_i$, contradicting the facts that x is nonadjacent to $b_2 \in B_i$ and B_i is a clique. If w_2 were adjacent to t, then, by definition of B_i, $t \in B_i$, contradicting $t \in T_i$. Therefore, w_1 is nonadjacent to x and w_2 is nonadjacent to t. Notice that w_1 is nonadjacent to b_2, since otherwise $\{w_1, b_1, b_2, x, v_{i+1}, t\}$ would induce tent in G. Also w_2 is nonadjacent to b_1, since otherwise $\{w_2, b_1, b_2, x, v_{i+1}, t\}$ would induce tent in G. Consequently, $w_1 \neq w_2$. Moreover, w_1 and w_2 are nonadjacent, since otherwise $\{v_i, b_1, b_2, w_2, w_1\}$ would induce 4-wheel in G. Recall that, by Claim 9, b_1 and b_2 are nonadjacent to v_{i+2}. Notice that at least one of w_1 and w_2 is adjacent to

v_{i+2}, since otherwise $\{b_1, b_2, v_{i+1}, w_1, w_2, v_{i+2}\}$ would induce net in G. If both w_1 and w_2 were adjacent to v_{i+2}, then $\{v_i, v_{i+1}, v_{i+2}, w_1, w_2\}$ would induce G_1 in G. If w_1 were adjacent to v_{i+2} but w_2 were nonadjacent to v_{i+2}, then $\{b_1, w_1, v_{i+2}, v_{i+1}, w_2\}$ would induce C_4^* in G. If w_2 were adjacent to v_{i+2} but w_1 were nonadjacent to v_{i+2}, then $\{b_2, w_2, v_{i+2}, v_{i+1}, w_1\}$ would induce C_4^*. As these contradictions arose from assuming the existence of P, the proof of the claim is complete. $\square$

Claim 11. *For each $b \in B_i$ and each $a \in A_{i+1}$, there is no chordless a, b-path in $G[B_i \cup T_i \cup A_{i+1}]$ together with a vertex $v \in A_{i+1} \cup T_i \cup B_i$ such that v is adjacent to no vertex of the path.*

Proof. Suppose, by the way of contradiction, that the claim is false and let $P = x_0 x_1 \ldots x_p$ be a chordless path in $G[B_i \cup O_i \cup A_{i+1}]$ with minimum number of vertices such that $x_0 \in B_i$, $x_p \in A_{i+1}$, and there is a vertex $v \in B_i \cup O_i \cup A_{i+1}$ which is adjacent to no vertex of P. Reasoning in a way entirely analogous to that employed in the beginning of the proof of Claim 6, it follows that x_0 is the only vertex of P in B_i, x_p is the only vertex of P in A_{i+1}, and $v, x_1, x_2, \ldots, x_{p-1} \in T_i$. Let $b = x_0$, $a = x_p$, and $t = v$. As $b \in B_i$ and $a \in A_{i+1}$, vertex b has some neighbor w_1 adjacent to v_i and non-adjacent to v_{i-1} and a has some neighbor adjacent to v_{i+1} and nonadjacent to v_i. By Claim 9, w_1 is nonadjacent to t and, symmetrically, w_2 is nonadjacent to t. Moreover, b is nonadjacent to w_2, since otherwise either $\{b, v_i, v_{i+1}, w_2, w_1\}$ would induce 4-wheel in G or $\{w_1, b, w_2, v_i, v_{i+1}, t\}$ would induce tent in G, depending on whether w_1 is adjacent to w_2 or not, respectively. Symmetrically, a is nonadjacent to w_1. In particular, $a \neq b$ and $p \geq 1$. Notice that, as $x_1, x_2, \ldots, x_{p-1} \in T_i$, both w_1 and w_2 are nonadjacent to each interior vertex P. We conclude that either $V(P) \cup \{t, w_1, w_2\}$ induces C_{p+3}^* in G or $V(P) \cup \{t, v_i, v_{i+1}, w_1, w_2\}$ induces $(p+3)$-tent in G, depending on whether w_1 and w_2 are adjacent or not, respectively. These contradictions show that the claim must be true. $\square$

Claim 12. *Each vertex of $B_i \cup A_{i+1}$ is an end vertex in $G[B_i \cup T_i \cup A_{i+1}]$.*

Proof. By symmetry, it suffices to prove that each vertex of B_i is an end vertex of $G[B_i \cup T_i \cup A_{i+1}]$. Suppose, by the way of contradiction, that not every vertex of B_i is an end vertex of $G[B_i \cup T_i \cup A_{i+1}]$. By Theorem 2, there is some minimum nonnegative integer s such that $G[B_i \cup T_i \cup A_{i+1}]$ contains an induced F_s (see Figure 3) where the filled vertex is some $b \in B_i$. Since $b \in B_i$, there is some neighbor w of b such that w is adjacent to v_i and nonadjacent to v_{i+1}.

Assume first that $s = 0$; i.e., there is a chordless path $P = x_1 x_2 x_3 x_4 x_5$ in $G[B_i \cup T_i \cup A_{i+1}]$ where $x_3 = b$. Reasoning in a way entirely analogous to that of case $s = 0$ of Claim 7 (replacing Claims 5 and 6 with Claims 10 and 11), $x_1, x_2, x_4, x_5 \in T_i$ and, by definition of T_i, none of x_1, x_2, x_4, and x_5 is adjacent to w. Hence, $\{x_1, x_2, b, x_4, x_5, v_{i+1}, w\}$ induces umbrella in G, a contradiction. Thus, $s \neq 0$.

Assume now that $s = 1$; i.e., there is a chordless path $P = x_1 x_2 x_3 x_4 x_5$ in $G[B_i \cup T_i \cup A_{i+1}]$ such that the only neighbor of b in P is x_3. Since B_i and A_{i+1} are cliques, either x_1 or x_5 belongs to T_i. Without loss of generality, assume that $x_1 \in T_i$. Since x_3, x_4, and x_5 belong to the same component of $G[B_i \cup T_i \cup A_{i+1}] \setminus N_G[x_1]$ than $b \in B_i$, Claim 11 implies that none of x_3, x_4, and x_5 belongs to A_{i+1}. Notice that $x_3 \notin B_i$ (since otherwise s would be 0) and none of x_4 and x_5 belongs to B_i because both vertices are nonadjacent to b and B_i is a clique (Claim 4). Necessarily, $x_3, x_4, x_5 \in T_i$. Symmetrically, $x_5 \in T_i$ implies that $x_1, x_2 \in T_i$. Hence, $V(P) \subseteq T_i$ and, by definition of T_i, no vertex of P is adjacent to w. Consequently, $V(P) \cup \{b, w\}$ induces bipartite claw in G, a contradiction. This contradiction proves that $s \neq 1$.

Since $s \geq 2$, there are a chordless path $P = a x_2 \ldots x_s x_{s+1}$ in $G[B_i \cup T_i \cup A_{i+1}]$ and two vertices $y_1, y_2 \in B_i \cup T_i \cup A_{i+1}$ whose neighborhoods in $B_i \cup T_i \cup A_{i+1}$ are $\{y_2\}$ and $\{a, x_2, \ldots, x_s\}$, respectively. As B_i and A_{i+1} are cliques, at least one of y_1 and x_{s+1} belongs to T_i. Reasoning in a way entirely analogous to that of case $s \geq 2$ of Claim 7 (replacing Claim 6 with Claim 11), we can prove that $x_3, x_4, \ldots, x_{s+1}, y_1 \in T_i$ and $x_2, y_2 \in B_i \cup T_i$. Notice that at least one of x_2 and y_2 is adjacent to w, since otherwise $\{w, b, x_2, x_3, \ldots, x_{s+1}, y_1, y_2\}$ would induce s-net in G. If x_2 is adjacent to w but y_2 is nonadjacent to w, then either $\{w, x_2, x_3, \ldots, x_{s+1}, y_1, y_2\}$ induces $(s-1)$-net in G or $\{w, b, x_2, x_3, y_1, y_2, v_{i+1}\}$ induces 4-tent in G, depending on whether $s \geq 3$ or not, respectively. If y_2 is adjacent to w but x_2 is nonadjacent to w, then $\{w, b, x_2, x_3, \ldots, x_{s+1}, y_1, y_2, v_{i+1}\}$ induces $(s+2)$-

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tent in G. Finally, if both x_2 and y_2 are adjacent to w, then $\{w, x_2, x_3, \ldots, x_{s+1}, y_1, y_2, v_{i+1}\}$ induces $(s+1)$-tent in G. These contradictions arose from assuming that there was some vertex in B_i that was not an end vertex of $G[B_i \cup T_i \cup A_{i+1}]$. This completes the proof of the claim. $\square$

Claim 13. B_i, A_{i+1} *is a left-right pair of* $G[B_i \cup T_i \cup A_{i+1}]$.

Proof. If follows from Theorem 3 and Claims 3, 4, and 10 to 12. $\square$

Claim 14. *If u and v are two adjacent vertices in $V(G) \setminus V(C)$, then u and v have at least one common neighbor in $V(C)$.*

Proof. Suppose, by the way of contradiction, that there are two adjacent vertices $u, v \in V(G) \setminus V(C)$ having no common neighbor in $V(C)$. By Theorem 4, $N_G(u) \cap V(C) = \{v_i, v_{i+1}, \ldots, v_j\}$ and $N_G(v) \cap V(C) = \{v_k, v_{k+1}, \ldots, v_\ell\}$, for some $i, j, k, \ell \in \{1, \ldots, n\}$. If $|N_G(u) \cap V(C)| \geq 3$, then $C' = v_i u v_j v_{j+1} \ldots v_{i-1} v_i$ would be a chordless cycle and the neighbors of v in $V(C')$ would contradict Theorem 4. Hence, $v_j = v_i$ or $v_j = v_{i+1}$. Symmetrically, $v_\ell = v_k$ or $v_\ell = v_{k+1}$. Let $P^1 = v_j v_{j+1} \ldots v_k$, $P^2 = v_\ell v_{\ell+1} \ldots v_i$. Without loss of generality, $|V(P^1)| \leq |V(P^2)|$. Let $m = |V(P^1)| + 2$; hence, $m \geq 4$.

Suppose first that $v_j = v_i$ and $v_\ell = v_k$. Then, $|V(P^2)| \leq 4$, since otherwise $V(P^1) \cup \{u, v, v_{k+2}\}$ would induce C_m^* in G. If $|V(P^1)| = |V(P^2)| = 4$, then $(V(C) \setminus \{v_k\}) \cup \{u, v\}$ would induce bipartite claw in G. Therefore, $V(C) \cup \{u, v\}$ induces domino, G_2, or G_4, depending on whether $(|V(P^1)|, |V(P^2)|) = (2, 4), (3, 3),$ or $(3, 4)$, respectively, a contradiction. Thus, $v_j \neq v_i$ or $v_\ell \neq v_k$.

Suppose now that $v_j = v_{i+1}$ and $v_\ell = v_k$. Then, $|V(P^2)| \leq 3$, since otherwise $V(P^1) \cup \{u, v, v_{k+2}\}$ would induce C_m^* in G. If $|V(P^1)| = |V(P^2)| = 3$, then $(V(C) \setminus \{v_k\}) \cup \{u, v\}$ would induce net in G. Necessarily, $(|V(P^1)|, |V(P^2)|) = (1, 2)$ and $V(C) \cup \{u, v\}$ induces G_3 in G, a contradiction.

Up to symmetry, it only remains to consider the case $v_j = v_{i+1}$ and $v_\ell = v_{k+1}$. On the one hand, if $|V(P^2)| \geq 3$, then $V(P_1) \cup \{u, v, v_{k+2}\}$ induces C_m^* in G. On the other hand, if $|V(P^1)| = |V(P^2)| = 2$, then $V(C) \cup \{u, v\}$ induces $\overline{C_6}$ in G. These contradictions complete the proof of the claim. $\square$

Claim 15. *If $o \in O_i$, then $N_G(o) = \{v_i\} \cup N_{G[A_i \cup O_i \cup B_i]}(o)$.*

Proof. Let $o \in O_i$. That $\{v_i\} \cup N_{G[A_i \cup O_i \cup B_i]}(o) \subseteq N_G(o)$ follows easily by definition of O_i and by definition of induced subgraphs. In order to prove the reverse inclusion, let $v \in N_G(o)$. If $v \in V(C)$, then $v = v_i$ by definition of O_i; so, assume that $v \notin V(C)$. Since $v \notin V(C)$, we deduce from $ov \in E(G)$ and Claim 14 that $v_i \in N_G(v)$. Hence, Theorem 4 ensures that $N_G(v) \cap V(C) = \{v_i\}$, $\{v_{i-1}, v_i\} \subseteq N_G(v)$, or $\{v_i, v_{i+1}\} \subseteq N_G(v)$ and, consequently, $v \in O_i$, $v \in A_i$, or $v \in B_i$, respectively (where o plays the role of w in proving $v \in A_i$ or $v \in B_i$). Thus, $N_G(o) \subseteq \{v_i\} \cup A_i \cup O_i \cup B_i$. Since $o \in O_i$, this is equivalent to $N_G(o) \subseteq \{v_i\} \cup N_{G[A_i \cup O_i \cup B_i]}(o)$, concluding the proof of the claim. $\square$

Claim 16. *If $t \in T_i$, then $N_G(t) = \{v_i, v_{i+1}\} \cup N_{G[B_i \cup T_i \cup A_{i+1}]}(t)$.*

Proof. Let $t \in T_i$. That $\{v_i, v_{i+1}\} \cup N_{G[B_i \cup T_i \cup A_{i+1}]} \subseteq N_G(t)$ follows immediately by definition of T_i and of induced subgraphs; so, we only need to prove the reverse inclusion. Let $v \in N_G(t)$. If $v \in V(C)$, then $v = v_i$ or $v = v_{i+1}$ by definition of T_i; so, assume that $v \notin V(C)$. From Claim 14, Theorem 4, and the definition of T_i, it follows that $N_G(v) \cap V(C)$ contains at least one of the following subsets $\{v_i, v_{i+1}\}$, $\{v_{i-1}, v_i\}$, or $\{v_{i+1}, v_{i+2}\}$. If $N_G(v) \cap V(C)$ does not contain $\{v_i, v_{i+1}\}$ but one of $\{v_{i-1}, v_i\}$ and $\{v_{i+1}, v_{i+2}\}$, then $v \in B_i$ or $v \in A_{i+1}$, respectively (where the role of w in proving $v \in B_i$ or $v \in A_{i+1}$ is played by t), contradicting $t \in T_i$. This contradiction shows that necessarily $\{v_i, v_{i-1}\} \subseteq N_G(v)$ and, consequently, $v \in A_i \cup T_i \cup B_i$ (as shown in the proof of Claim 1); thus, since $t \in T_i$, $N_G(t) \subseteq \{v_i, v_{i+1}\} \cup N_{G[B_i \cup T_i \cup A_{i+1}]}$, which completes the proof of the claim. $\square$

Claim 17. $N_G(v_i) \setminus V(C) = B_{i-1} \cup T_{i-1} \cup A_i \cup O_i \cup B_i \cup T_i \cup A_{i+1}$.

Proof. That $B_{i-1} \cup T_{i-1} \cup A_i \cup O_i \cup B_i \cup T_i \cup A_{i+1} \subseteq N_G(v_i) \setminus V(C)$ follows by definition of the respective sets. In order to prove the reverse inclusion, let $v \in N_G(v_i) \setminus V(C)$. Since $v_i \in N_G(v)$, Theorem 4 implies that either $N_G(v) \cap V(C) = \{v_i\}$ or $N_G(v) \cap V(C)$ contains $\{v_{i-1}, v_i\}$ or $\{v_i, v_{i+1}\}$ as subsets. On the one hand, if $N_G(v) \cap V(C) = \{v_i\}$, then $v_i \in O_i$ by definition. On the other hand, if $N_G(v) \cap V(C)$ contains $\{v_{i-1}, v_i\}$ or $\{v_i, v_{i+1}\}$ as subsets, then $v \in B_{i-1} \cup T_{i-1} \cup A_i$ or $v \in B_i \cup T_i \cup A_{i+1}$, respectively (as shown in the proof of Claim 1). We conclude that also the reverse inclusion $N_G(v_i) \setminus V(C) \subseteq B_{i-1} \cup T_{i-1} \cup A_i \cup O_i \cup B_i \cup T_i \cup A_{i+1}$ holds and the claim is proved. $\square$

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Constructing a circular-arc model for G**.** Consider a circular-arc model $\mathcal{M}$ of the chordless cycle C where $\mathcal{M}$ consists of a set of closed arcs on a circle $\mathscr{C}$ such that no two arcs of $\mathcal{M}$ share a common endpoint. For each arc X on $\mathscr{C}$, we call *left* and *right* endpoints to the starting and ending endpoints of X, respectively, when traversing $\mathscr{C}$ in clockwise direction. For each $j \in \{1,\ldots,n\}$, let $X(v_j)$ be the arc of $\mathcal{M}$ corresponding to v_j and let ℓ_j and r_j be the left and right endpoints of $X(v_j)$, respectively. For each $i \in \{1,\ldots,n\}$, we use Claim 8 to build an intersection model $\mathcal{M}_i^1$ of $G[A_i \cup O_i \cup B_{i+1}]$ consisting of open arcs within the clockwise open arc (r_{i-1}, ℓ_{i+1}) on $\mathscr{C}$ such that the left endpoint of each arc corresponding to a vertex in A_i is r_{i-1} and the right endpoint of each arc corresponding to a vertex in B_i is ℓ_{i+1}. For each $v \in A_i \cup O_i \cup B_i$, let $X_i^1(v)$ be the arc representing v in $\mathcal{M}_i^1$. Analogously, for each $i \in \{1,\ldots,n\}$, we use Claim 13 to build an intersection model $\mathcal{M}_i^2$ of $G[B_i \cup T_i \cup A_{i+1}]$ consisting of closed arcs within the clockwise closed arc $[\ell_{i+1}, r_i]$ on $\mathscr{C}$ such that the left endpoint of each arc corresponding to a vertex in B_i is ℓ_{i+1} and the right endpoint of each arc corresponding to a vertex in A_{i+1} is r_i. For each $v \in B_i \cup T_i \cup A_{i+1}$, let $X_i^2(v)$ be the arc representing v in $\mathcal{M}_i^2$. For each $v \in V(G) \setminus V(C)$, we define $X(v)$ as the union of all the arcs corresponding to v among the models $\mathcal{M}_1^1,\ldots,\mathcal{M}_n^1,\mathcal{M}_1^2,\ldots,\mathcal{M}_n^2$; i.e., $X(v) = \bigcup_{i: v \in A_i \cup O_i \cup B_i} X_i^1(v) \cup \bigcup_{i: v \in B_i \cup T_i \cup A_{i+1}} X_i^2(v)$.

Claims 18 and 19 prove that the sets $X(v)$ (as defined in the preceding paragraph) are arcs on $\mathscr{C}$ and that the family consisting of all the arcs $X(v)$ is a circular-arc model for G, respectively.

Claim 18. *For each* $v \in V(G)$, $X(v)$ *is an arc on* $\mathscr{C}$.

Proof. If $v \in V(C)$, the claim is true by definition. Suppose first that $v \in O_i$. Then, since for every $j \in \{1,\ldots,n\}$ each vertex of $A_j \cup O_j \cup B_j$ is adjacent to v_j and each vertex of $B_j \cup T_j \cup A_{j+1}$ is adjacent to v_j and v_{j+1}, then $v \in A_j \cup O_j \cup B_j$ only when $j = i$, and $v \notin B_j \cup T_j \cup A_{j+1}$ for each $j \in \{1,\ldots,n\}$; therefore, by definition, $X(v) = X_i^1(v)$. Suppose now that $v \in T_i$. Then, since $v \in B_j \cup T_j \cup A_{j+1}$ only when $j = i$ and $v \notin B_j \cup O_j \cup A_j$ for each $j \in \{1,\ldots,n\}$, the equality $X(v) = X_i^2(v)$ holds. So, assume that $v \in \bigcup_{i=1}^n (A_i \cup B_i)$ and it only remains to show that $X(v)$ is an arc. By Theorem 4, the neighbors of v in $V(C)$ induce a chordless path $P = v_j v_{j+1}\ldots v_k$ for some $j,k \in \{1,\ldots,n\}$ and $j \neq k$. By definition, $v \notin A_i \cup O_i \cup B_i$ for each $i \notin \{j, j+1,\ldots,k\}$ and $v \notin B_i \cup T_i \cup A_{i+1}$ for each $i \notin \{j, j+1,\ldots,k-1\}$. As $v \in A_{j+1}, B_{j+1},\ldots,A_{k-1}, B_{k-1}$ by definition, the sets $X_{j+1}^1(v), X_{j+1}^2(v), \ldots, X_{k-2}^2(v), X_{k-1}^1(v)$ are the clockwise arcs (r_j, ℓ_{j+2}), $[\ell_{j+2}, r_{j+1}]$, $\ldots$, $[\ell_{k-1}, r_{k-2}]$, (r_{k-2}, ℓ_k) on $\mathscr{C}$, respectively, whose union is the clockwise open arc (r_j, ℓ_k) on $\mathscr{C}$. If there is a neighbor w_L of v adjacent to v_j and nonadjacent to v_{j+1}, then $v \in B_j$ and let $X_L(v) = X_j^1(v)$; otherwise, $v \notin A_j \cup O_j \cup B_j$ and let $X_L(v) = \emptyset$. If there is a neighbor w_R of v adjacent to v_k and nonadjacent to v_{k-1}, then $v \in A_k$ and let $X_R(v) = X_k^1(v)$; otherwise, $v \notin A_k \cup O_k \cup B_k$ and let $X_R(v) = \emptyset$. Finally, $X(v) = X_L(v) \cup X_j^2(v) \cup (r_j, \ell_k) \cup X_{k-1}^2(v) \cup X_R(v)$, which is an arc on $\mathscr{C}$ because, whenever $X_L(v) \neq \emptyset$, the right endpoint of $X_L(v)$ and the left endpoint of $X_j^2(v)$ are ℓ_{j+1} and, whenever $X_R(v) \neq \emptyset$, the right endpoint of $X_{k-1}^2(v)$ and the left endpoint of $X_R(v)$ are r_{k-1}. The proof of the claim is complete. $\square$

Claim 19. *The family consisting of all the arcs* $X(v)$ *for each* $v \in V(G)$ *is circular-arc model for* G.

Proof. Let $\tilde{G}$ be the graph having the same set of vertices than G and such that two vertices u and v of $\tilde{G}$ are adjacent if and only if $X(u)$ and $X(v)$ have nonempty intersection. The claim is proved as soon as we verify that $\tilde{G} = G$. Let $o \in O_j$ for some $j \in \{1,\ldots,n\}$. In the proof of Claim 18 we showed that $X(o) = X_j^1(o)$ which, by construction, intersects precisely the arc $X(v_j)$ and the arcs $X(v)$ for each each neighbor v of o in $A_j \cup O_j \cup B_j$. As a result and by Claim 15, $N_{\tilde{G}}(o) = N_G(o)$. Let $t \in T_j$ for some $j \in \{1,\ldots,n\}$. In the proof of Claim 18 we showed that $X(t) = X_j^2(t)$ which, by construction, intersects precisely the arcs $X(v_j)$ and $X(v_{j+1})$ and the arcs $X(v)$ for each neighbor of t in $B_j \cup T_j \cup A_{j+1}$. Consequently, by Claim 16, $N_{\tilde{G}}(t) = N_G(t)$. Notice that, also by construction, for each $j \in \{1,\ldots,n\}$, the arc $X(v_j)$ intersects precisely the arcs $X(v_{j-1})$, $X(v_{j+1})$, and the arcs $X(v)$ for every $v \in B_{j-1} \cup T_{j-1} \cup A_i \cup O_i \cup B_i \cup T_i \cup A_{j+1}$, which, by Claim 17, means that $N_{\tilde{G}}(v_j) = N_G(v_j)$. In order to complete the proof of the claim, let $u, v \in \bigcup_{i=1}^n (A_i \cup B_i)$ and it only remains to prove that $uv \in E(\tilde{G})$ if and only if $uv \in E(G)$. Since $u, v \notin V(C)$, the construction ensures that if

$uv \in E(\tilde{G})$ then $uv \in E(G)$. For the converse, assume that $uv \in E(G)$. By Claim 14 and Theorem 4, u and v have at least one common neighbor in $V(C)$ and the common neighbors of u and v in $V(C)$ induce a chordless path P. If $|V(P)| \geq 2$, then $\{v_j, v_{j+1}\} \subseteq V(P)$ for some $j \in \{1, \ldots, n\}$ and, consequently, $u, v \in B_j \cup T_j \cup A_{j+1}$ (as shown in the proof of Claim 1); thus, $uv \in E(G[B_j \cup T_j \cup A_{j+1}])$ and, by construction, $uv \in E(\tilde{G})$. Finally, let us consider the case $V(P) = \{v_j\}$ for some $j \in \{1, \ldots, n\}$. As $u \notin O_j$ and $v \notin O_j$, then, up to symmetry, $N_G(u) \cap \{v_{j-1}, v_j, v_{j+1}\} = \{v_{j-1}, v_j\}$ and $N_G(v) \cap \{v_{j-1}, v_j, v_{j+1}\} = \{v_j, v_{j+1}\}$. Hence, $u \in A_j$ (with v playing the role of w in the definition) and $v \in B_j$ (with u playing the role of w in the definition); thus, $uv \in E(G[A_j \cup O_j \cup B_j])$ and construction implies that $uv \in E(\tilde{G})$. This completes the proof of the claim. $\qquad\square$

By Claim 19, G is a circular-arc graph and, as a result, Theorem 5 implies that G is a normal Helly circular-arc graph. This completes the proof of Theorem 6. $\blacksquare$

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