Incidence matrices for oriented graphs and applications

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Let G be a simple graph and $\mathcal{L}(G)$ its line graph. In this work, if an arbitrary orientation is given to each edge of G, we obtain a digraph G' and introduce the notions of incidence and coincidence between its edges. Using these notions, we considere two graphs $\mathcal{L}_0(G')$ and $\mathcal{L}_1(G')$, called *incidence-line graph* and *coincidence-line graph* of G', respectively, whose adjacency matrices satisfy $A_{\mathcal{L}}(G) = A_{\mathcal{L}_0}(G') + A_{\mathcal{L}_1}(G')$. Let $K(G')$ be the $(-1, 0, 1)$-incidence matrix of G'. It is well known (see [2, 3]) that the matrix $K(G')$ satisfies the identity:

$$K(G')\, K(G')^t = \mathrm{D}(G) - A(G) = L(G), \tag{1}$$

where $L(G)$ is the Laplacian matrix of G.

We prove that

$$K(G')^t\, K(G') = 2\mathrm{I}_m + A_{\mathcal{L}_0}(G') - A_{\mathcal{L}_1}(G') \tag{2}$$

and conclude that the matrices $A_{\mathcal{L}_0}(G')$ and $A_{\mathcal{L}_1}(G')$ are associated to the Laplacian matrix of G.

Using this fact, we present a lower bound for the largest Laplacian eigenvalue of any triangle-free graph. Finally, we obtain new form for the Laplacian and signless Laplacian energies of any graph and a relation between both energies is derived.

Keywords: Graphs, digraphs, line graph, incidence matrix, adjacency matrix, Laplacian matrix, Laplacian energy, signless Laplacian energy.

References

[1] Gutman, I., Kiani, D., Mirzakhah, M., Zhou, B. On incidence energy of a graph. *Lin. Algebra and its Applications* **431** (2009) 1223–1233

[2] Merris, R. Laplacian matrices of a graph: A survey. *Linear Algebra and its Applications* **197-198** (1994) 143-176

[3] Merris, R. A survey of graph Laplacians, *Linear and Multilinear Algebra* **39** (1995) 19-31.