CONSEQUENCES OF A NEW UPPER BOUND ON THE SUM OF THE LARGEST LAPLACIAN EIGENVALUES OF TREES

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In this work, we use an upper bound on the sum of the Laplacian eigenvalues of trees recently obtained by the current authors [2] to address two problems related with the Laplacian spectrum: the quest for the set of trees with largest Laplacian energy and Brouwer's Conjecture.

Given a graph G on the vertex set $V = \{v_1, \dots, v_n\}$, the *Laplacian matrix* of G is given by $L = D - A$, where D is the diagonal matrix whose entry (i,i) is equal to the degree of v_i and A is the adjacency matrix of G. The *Laplacian spectrum* of G is defined as the set of eigenvalues of L, which we shall denote $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$. With this, the *Laplacian energy* of G is given by $LE(G) = \sum_{i=1}^{n} |\mu_i - \overline{d}|$, where $\overline{d}$ is the average degree of G.

A total ordering of graphs by the Laplacian energy is known in the case of n-vertex trees whose diameter is equal to three. A tree in this class can be viewed as the union of two stars with an edge between their centers. The endpoints of this edge are called the two ends of the tree. Every tree in thus class may be written in the form $T(a,b)$, where $n = a+b+2$ is the number of vertices and $a, b \geq 1$ determine the number of leaves incident with each end of $T(a,b)$. For fixed integers $a \geq b \geq 1$, consider the class $\mathcal{T}(a,b) = \{T(a+k, b-k) \colon k = 0, \dots, b-1\}$. Trevisan, Carvalho, Del-Vecchio and Vinagre [4] have shown that the Laplacian energy of the elements of $\mathcal{T}(a,b)$ decreases as k gets larger.

Here, we investigate the problem of ordering the full class of n-vertex trees according to their Laplacian energy. More precisely, for any fixed n, we find a list of roughly $\sqrt{n}$ trees on n vertices with largest Laplacian energy. With the exception of the n-vertex star, we prove that all trees in this class have diameter three, so that the actual order is inherited from the order on $\mathcal{T}(a,b)$. However, for every $n \geq 16$, we give a counterexample to the statement that trees with diameter three always have larger Laplacian energy than trees with larger diameter.

In particular, our work settles a conjecture of Radenković and Gutman [3] regarding the n-vertex trees with second, third and fourth largest Laplacian energy, which are, respectively, $T(\lceil (n-2)/2 \rceil, \lfloor (n-2)/2 \rfloor)$, $T(\lceil (n-2)/2 \rceil + 1, \lfloor (n-2)/2 \rfloor - 1)$ and $T(\lceil (n-2)/2 \rceil + 2, \lfloor (n-2)/2 \rfloor - 2)$. The main tool to prove this result is the new bound $S_k(T) \leq n - 2 + 2k - \frac{2k}{n}$ for every tree T with $n \geq 6$ vertices and diameter $d \geq 4$ [2].

We also deal with Brouwer's Conjecture [1], which states that, given a graph $G = (V, E)$ with n vertices and an integer $k \in \{1, \dots, n\}$, the sum $S_k(G)$ satisfies

$$S_k(G) \leq |E| + \binom{k+1}{2}. \tag{1}$$

The new upper bound mentioned above implies a new upper bound on $S_k(G)$ for n-vertex graphs with diameter at least four, which has the following consequence in terms of Brouwer's Conjecture.

Theorem 1. *Let G be a graph with $n \geq 6$ vertices and diameter $d \geq 4$. The inequality* (1) *is satisfied for*

$$\frac{3n - 4 + \sqrt{9n^2 - 24n + 16 + 8e(G)n^2 - 8n^3}}{2n} \leq k \leq n.$$

References

[1] A.E. Brouwer and W.H. Haemers, Spectra of graphs, Springer, 2012.
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[4] V. Trevisan, J. B. Carvalho, R. Del-Vecchio, and C. Vinagre, Laplacian energy of diameter 3 trees, 2011, Applied Mathematics Letters, v. 24, p. 918-923.