AN UPPER BOUND ON THE SUM OF THE LARGEST LAPLACIAN EIGENVALUES OF A TREE

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Given a graph G on the vertex set $V = \{v_1, \dots, v_n\}$, the *Laplacian matrix* of G is given by $L = D - A$, where D is the diagonal matrix whose entry (i,i) is equal to the degree of v_i and A is the adjacency matrix of G. The *Laplacian spectrum* of G is defined as the set of eigenvalues of L, which we shall denote $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$. For any given $k \in \{1, \dots, n\}$, we are interested in the sum $S_k(T) = \sum_{i=1}^{k} \mu_i$ of the k largest Laplacian eigenvalues of a tree T.

Brouwer's Conjecture states that, given a graph $G = (V,E)$ with n vertices, the sum $S_k(G)$ satisfies

$$S_k(G) \leq |E| + \binom{k+1}{2}.$$

Haemers, Mohammadian and Tayfeh-Rezaie [2] have proved this conjecture for all trees T and arbitrary values of k with the inequality $S_k(T) \leq n - 2 + 2k$. Fritscher et al. [1] have refined this result to $S_k(T) \leq n - 2 + 2k - \frac{2k-2}{n}$. This upper bound is tight when $k = 1$ and T is a star, and we have shown that it cannot be improved by subtracting $1/n$, even if we consider trees with diameter at least three. In this work, we show that this upper bound may be improved by $2/n$ for trees with diameter at least four.

Theorem 1. *Every tree T with $n \geq 6$ vertices and diameter greater than or equal to 4 satisfies*

$$S_k(T) \leq n - 2 + 2k - \frac{2k}{n}.$$

To do this, we use three powerful tools. We use a characterization by Rojo [4] to decompose the spectrum of a large tree into spectra os smaller matrices. A classical result by Wielandt [5] tells us that, to find an upper bound on the sum of given set of eigenvalues of A, it suffices to look at the eigenvalues of matrices B and C such that $B + C = A$. Finally, we use an algorithm by Jacobs et al. [3] (adapted in [1]) to locate the number of eigenvalues of a tree that are larger than any given constant α.

References

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