Some results on nonsingular acyclic matrices with an extremal number of P-vertices

An important connection between graphs and matrices associates to any undirected graph G, the set $\mathcal{S}(G)$ consists of all real symmetric matrices $A = (a_{ij})$ such that, for $i \neq j$, $a_{ij} = 0$ if and only if ij is not an edge of G, where no constraint is placed upon the diagonal entries of the matrices determined by the graph. Such matrices can be seen as weighted adjacency matrices of G. If G is a tree, these matrices are called *acyclic*. Reciprocally, given a real symmetric matrix $A = (a_{ij})$ of order n, the *graph of* A, denoted by $\mathcal{G}(A)$, is the graph whose vertex set is $\{1, 2, \ldots, n\}$ and edge set is $\{ij \mid i \neq j \text{ and } a_{ij} \neq 0\}$. Note that the main diagonal of A has no role in determining $\mathcal{G}(A)$. Therefore

$$\mathcal{S}(G) = \{A \in \mathbb{R}^{n\times n} \mid A \text{ is symmetric and } \mathcal{G}(A) = G\}$$

and an analogous association can be made between graphs and Hermitian matrices.

For $A \in \mathcal{S}(G)$, let $m_A(\lambda)$ denote the multiplicity of λ as a root of the characteristic polynomial of A. In the case of λ is not an eigenvalue of A we write $m_A(\lambda) = 0$. If S is an index subset of $\{1, 2, \ldots, n\}$, then we denote by $A(S)$ the principal submatrix of A resulting from the deletion of the rows and columns indexed by S. In particular, when S consists of a single index, say i, we simply write $A(i)$ for $A(\{i\})$.

As a consequence of the Cauchy Interlacing Theorem, we know that

$$\left| m_{A(i)}(0) - m_A(0) \right| \leqslant 1\,.$$

In the case of $m_{A(i)}(0) = m_A(0) + 1$, the index i is called a *P-vertex* of A.

The study of the P-vertices of a symmetric matrix has been the object of a fresh and prosperous research.

It is known that for any nonsingular acyclic matrix of order n, the maximum number of P-vertices is n if n is even, and $n - 1$ if n is odd. In this talk, we thoroughly characterize the trees where those bounds are achieved. In addition, for those trees and for any nonnegative integer k less than or equal to the extremal number of P-vertices, we provide an algorithm to construct a nonsingular matrix whose graph is the given tree and the number of P-vertices is k. Illustrative examples will be given.

This is a joint work with Zhibin Du.