Efficient Computation of the Characteristic Polynomial of a Graph of Bounded Tree-Width

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Abstract

An $O(n \log^2 n)$ algorithm is presented to compute the characteristic polynomial of a graph of bounded tree-width on n vertices, improving on the previously best time $O(n^{2.376})$ (matrix multiplication time). Previously, such a fast algorithm was known for trees [Für09] based on a technique combining divide-and-conquer with dynamic programming.

For any (undirected, loop-less) graph with adjacency matrix A, elementary considerations of the characteristic polynomial

$$\chi(A;\lambda) = \det(\lambda I - A) = \sum_{i=0}^{n} c_i \lambda^{n-i}$$

show the well known result that

$$c_r = \sum_{\sigma} (-1)^r \mathrm{sgn}(\sigma)$$

where σ ranges over all directed (simple) cycle packings covering r vertices (see Biggs [Big93]). Each such directed cycle packing can be identified with a permutation σ of the n vertices with exactly $n-r$ fixed points, and with $a_{v\sigma(v)} = 1$ (meaning $\{v, \sigma(v)\}$ is an edge) whenever v is not a fixed point of σ.

For trees the only directed cycle packings consist of matchings (a collection of cycles of length 2 and fixed-points). As a consequence, the characteristic polynomial of a tree is equal to the matching polynomial.

The algorithm for computing the matching polynomial can be extended naturally from trees to graphs of bounded tree-width. To count the numbers of matchings, it is sufficient to count such matching iteratively for every boundary condition. This does not easily extend to count directed cycle packings with the proper sign, because one has also to count the number of cycles inside. A straight-forward extension to bounded tree-width would loose a factor of n in the running time.

References

[Big93] N. Biggs, *Algebraic graph theory*, second ed., Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1993.

[Für09] Martin Fürer, *Efficient computation of the characteristic polynomial of a tree and related tasks*, ESA, LNCS, vol. 5757, Springer, 2009, pp. 11–22.