# Odd Graphs are Prism-Hamiltonian and Have a Long Cycle

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## ABSTRACT

The odd graph $O_k$ has the subsets with $k$ elements of a set $\{1, \ldots, 2k+1\}$ as its vertices set, and there exists an edge between two vertices if the corresponding pair of $k$-subsets is disjoint. A conjecture claims that $O_k$ is hamiltonian for $k > 2$ and another long-standing conjecture implies that all odd graphs have a hamiltonian path. We proved that the prism over $O_k$ is hamiltonian and that $O_k$ has a cycle with at least $.625|V(O_k)|$ vertices.

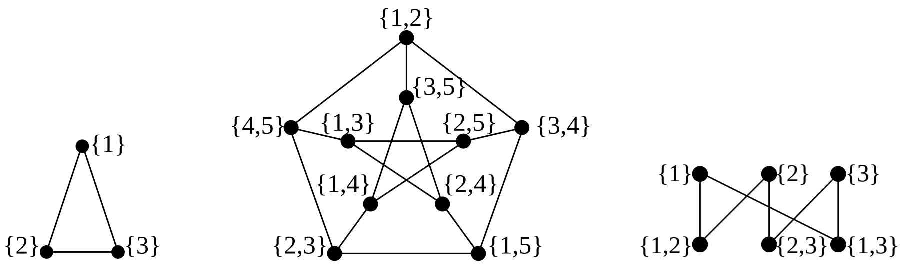
**KEYWORDS. Hamiltonian Cycle, Hamiltonian Path, Odd Graph.**

**Main area: TAG - Graph Theory and Algorithms in Graphs.**

## 1. Introduction

A long-standing conjecture due to Lovász claims that every connected undirected vertex-transitive graph has a hamiltonian path [Lovász(1970)]. Since the odd graphs form a family of connected vertex-transitive graphs, they can provide a counterexample or more evidence to support Lovász' conjecture. Later, [Biggs(1979)] conjectured that the odd graphs are hamiltonian for all $k > 2$. Still, a related conjecture by [Havel(1983)] claims that the bipartite double graph of the odd graph is hamiltonian.

The vertices of the *odd graph* $O_k$ are the $k$-subsets of $\{1, 2, \ldots, n = 2k + 1\}$ and two vertices are adjacent if the corresponding $k$-subsets are disjoint (Figures 1(a) and 1(b)). The bipartite double graph of the odd graph $O_k$ is called the *middle-layers graph* and it is denoted by $B_k$. The vertices are the $k$-subsets and $(k + 1)$-subsets of $\{1, 2, \ldots, n = 2k + 1\}$ and the edges represent the inclusion between two such subsets (Figure 1(c)). The vertex set of $B_k$ can be seen as the two middle layers of the $n$-dimensional cube.




(a) $O_1$ (b) $O_2$ (c) $B_1$

Figure 1: The odd graph $O_k$ for $k = 1, 2$ and the middle-levels graph $B_k$ for $k = 1$.

At the moment, Biggs' conjecture has been verified only for some values of $k$. The odd graph is hamiltonian for $3 \leq k \leq 13$ [Shields and Savage(2004)], and has a hamiltonian path for $k \leq 19$ [Bueno et al.(2009), Shimada and Amano(2011)]. On the other hand, Havel's conjecture is true for $k \leq 19$ [Shields et al.(2009), Shimada and Amano(2011)].

Since the decision problem of the hamiltonian cycle problem is NP-Complete [Karp(1972)], one recent trend is to search for long cycles. [Johnson(2004)] provided a lower bound showing that $O_k$ and $B_k$ contain a cycle of length $(1 - o(1))|V(O_k)|$ and $(1-o(1))|V(B_k)|$, respectively, where the error term $o(1)$ is of the form $\frac{c}{\sqrt{k}}$ for some constant $c$. Although the author does not estimate $c$, this means that $O_k$ and $B_k$ are asymptotically hamiltonian. [Savage and Winkler(1995)] showed that if $B_k$ has a hamiltonian cycle for $k \leq h$, then $B_k$ has a cycle containing a fraction $1 - \varepsilon$ of the graph vertices for all $k$, where $\varepsilon$ is a function of $h$. For example, since $B_k$ has a hamiltonian cycle for $1 \leq k \leq 19$, the graph $B_k$ has a cycle containing at least 87.46% of the graph vertices for $k \geq 20$. For the odd graph $O_k$, the best lower bound known on the length of the longest cycle is $\sqrt{3|V(O_k)|}$, given by [Babai(1979)] for vertex-transitive graphs in general, which is less than 3% for $O_{10}$, and asymptotically approaches zero as $k$ increases.

Another trend is to search for related structures and, in this aspect, having a hamiltonian prism in a graph has been shown to be an interesting relaxation of being hamiltonian [Kaiser et al.(2007)]. The *prism over a graph* $G$ is the Cartesian product $G \square K_2$ of $G$ with the complete graph on two vertices (Figure 2). If the prism over $G$ is hamiltonian, then $G$ is *prism-hamiltonian.*

[Horák et al.(2005)] established that the prism over $B_k$ is hamiltonian and, later, the counterpart of this result was proved for $O_k$, but only for $k$ even [Bueno and Horák(2011)]. In our work, published in [Mesquita et al.(2014)], we demonstrated that the prism over $O_k$ is hamiltonian for all $k$. Moreover, we improve the lower bound on the length of the longest cycle of $O_k$ by providing a cycle with $.625|V(O_k)|$ vertices at least.

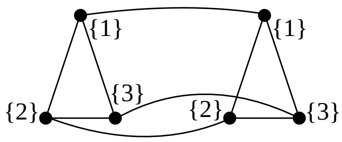



Figure 2: The prism over the graph $O_1$.

In Section 2, we give some auxiliary results to discuss our main results in Section 3. Section 4 details the participation of the undergraduate student in this research. The present text is meant to be a brief introduction to the basic ideas underlying the proofs contained in the papers [Mesquita and Bueno(2013), Mesquita et al.(2014)]. Obviously, it does not delve too much into details due to space constraints.

## 2. Preliminaries

A *spanning cactus* in a graph $G$ is a spanning connected subgraph of maximum degree 3 consisting of vertex-disjoint cycles and vertex-disjoint paths. The cactus is *even* if all cycles are even.

**Proposition 1** ([Čada et al.(2004)]). *If $G$ contains a spanning even cactus, then $G$ is prism-hamiltonian.*

The main idea of our results consists in showing that $O_k$ contains a spanning even cactus consisting of an even cycle and paths of size 1 and/or 2 connected to the cycle. Additionally, we show that the even cycle has at least $.625|V(O_k)|$ vertices. First, we discuss some definitions and auxiliary results.

There is a correspondence between the $k$-subsets and the $(n-k)$-subsets of $\{1, 2, \dots, n = 2k+1\}$ with a set of binary strings of $n$ bits with exactly $k$ 1's and $(n-k)$ 0's (Figure 3). The correspondence $b_n b_{n-1} \dots b_1 \to \{i | b_i = 1\}$ is a bijection of binary strings of $n$ bits into the subsets of $n$. The *complement* $\overline{x}$ of a binary digit $x$ is 1 if $x = 0$ and 0 if $x = 1$. The complement of a binary string extends this definition by bitwise complement. From now on, we consider the vertices of $O_k$ and $B_k$ represented by binary strings.

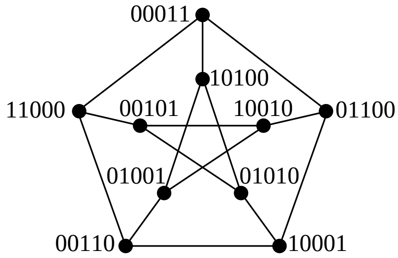



Figure 3: The odd graph $O_2$ from Figure 1(b) represented with the vertices as binary strings.

**Definition 2** ([Mesquita et al.(2014)]). *Let $C = (v_1, v_2, v_3, v_4, \dots, v_q)$ be a cycle or path in $O_{k-1}$. We define the sequences $C_1$ and $C_2$ as follows:*

*(i) If $q$ is even, then $C_1 = (0v_11, 1v_20, 0v_31, 1v_40, \dots, 1v_q0)$ and $C_2 = (1v_10, 0v_21, 1v_30, 0v_41, \dots, 0v_q1)$.*

*(ii) If $q$ is odd, then $C_1 = (0v_11, 1v_20, 0v_31, 1v_40, \dots, 0v_q1)$ and $C_2 = (1v_10, 0v_21, 1v_30, 0v_41, \dots, 1v_q0)$.*

Lemma 3 shows that, if $C$ is a cycle in $O_{k-1}$, then $C_1$ and $C_2$ are either one or two cycles in $O_k$.

Given two disjoint paths $Q_1$ and $Q_2$ such that the last vertex of $Q_1$ is adjacent to the first vertex of $Q_2$, denote by $Q_1 \circ Q_2$ the path obtained by traversing the vertices of $Q_1$, and then the vertices of $Q_2$.

**Lemma 3** ([Mesquita et al.(2014)]). *Let $C$ be a cycle with $q$ vertices in $O_{k-1}$.*

*(i) If $q$ is odd then there is a cycle with $2q$ vertices in $O_k$.*

*(ii) If $q$ is even then there are two disjoint cycles each with $q$ vertices in $O_k$.*

*Proof.* By adding 1 and 0 to a vertex of $C$ we have a vertex of $O_k$. Therefore, construct the paths $C_1$ and $C_2$ in $O_k$ according to Definition 2. By construction, $C_1$ and $C_2$ are paths in $O_k$ and, if $q$ is even, both are cycles as well. If $q$ is odd, since $C$ is a cycle, then there are edges $\{0v_1 1, 1v_q 0\}$ and $\{1v_1 0, 0v_q 1\}$ in $E(O_k)$, which results in a cycle $C_1 \circ C_2$ in $O_k$. □

**Definition 4** ([Mesquita et al.(2014)]). *Let $S_k$, $T_k$ and $R_k$ be three disjoint subsets of the vertices of $O_k$ such that:*

*(i) $S_k$ is the set of $k$-subsets which have element 1 or $n$, but not both.*

*(ii) $T_k$ is the set of $k$-subsets which neither has element 1 nor element $n$.*

*(iii) $R_k$ is the set of $k$-subsets which have both elements 1 and $n$.*

Notice that $V(O_k) = S_k \cup T_k \cup R_k$.

**Lemma 5** ([Mesquita et al.(2014)]). *Each vertex $v \in T_k$ has exactly two edges to vertices of $S_k$ and $(k-1)$ edges to vertices of $R_k$.*

By definition, the vertices of $S_k$ have no edges to vertices of $R_k$ and, by Lemma 5, they have exactly two edges to vertices of $T_k$ and $(k-1)$ edges to vertices of $S_k$. Clearly, the vertices of $R_k$ are adjacent only to vertices of $T_k$. Therefore, $O_k$ has a bipartite subgraph with bipartition $(T_k, R_k)$ such that the partition $T_k$ has degree $(k-1)$ and the partition $R_k$ has degree $(k+1)$.

**Lemma 6** ([Mesquita et al.(2014)]). *It holds that $|T_k| > |R_k|$.*

**Lemma 7** ([Mesquita and Bueno(2013), Mesquita et al.(2014)]). *If $C = (v_1, v_2, \ldots, v_q)$ is a hamiltonian cycle in $O_{k-1}$ and $C_1$ and $C_2$ are constructed according to Definition 2, then*

*(i) $S_k = V(C_1) \cup V(C_2)$;*

*(ii) $|S_k| = |C_1| + |C_2| = 2|V(O_{k-1})| > 0.5|V(O_k)|$.*

**Lemma 8** ([Mesquita et al.(2014)]). *It holds that $|T_k| = |V(O_{k-1})|$.*

**Theorem 9** ([Mesquita and Bueno(2013), Mesquita et al.(2014)]). *If there exists a hamiltonian cycle $C = (v_1, \ldots, v_q)$ in $O_{k-1}$, then $O_k$ has a cycle $C'$ such that $|C'| > 0.75|V(O_k)|$.*

*Proof.* Construct $C_1$ and $C_2$ according to Definition 2. Notice that there are $q$ vertices $0\overline{v_j}0$ connecting $0v_j1$ to $1v_j0$, where $0v_j1 \in S_k$, $1v_j0 \in S_k$ and $v_j \in C$ for $1 \leq j \leq q$, since the complement $\overline{v_j}$ of a vertex $v_j \in C$ has $k$ 1's and $(k-1)$ 0's. Therefore, $0\overline{v_j}0 \in V(O_k)$. Construct $q$ paths with 3 vertices by combining the vertices of $C_1$, $C_2$ and $T_k$:

$$Q_1 = 0v_1 1, 0\overline{v_1}0, 1v_1 0$$

$$Q_2 = 1v_20, 0\overline{v_2}0, 0v_21$$
$$\vdots$$
$$Q_q = 1v_q0, 0\overline{v_q}0, 0v_q1, \text{ if } q \text{ is even or}$$
$$Q_q = 0v_q1, 0\overline{v_q}0, 1v_q0, \text{ if } q \text{ is odd.}$$

For $Q_j$, $1 \le j \le q$, the first vertex of $Q_j$ is in $C_1$, the second one is in $T_k$ and the third one is in $C_2$. The $q$ vertices $0\overline{v_j}0$, for $v_j \in C$, are distinct, since $C$ is a hamiltonian cycle in $O_{k-1}$ and, therefore, the complement of the vertices of $C$ are distinct as well and, by Lemmas 8, consist of all vertices of $T_k$. Denote by $\overleftarrow{Q}$ a path $Q$ traversed from the last to the first vertex. Concatenate the $q$ paths $Q_j$, for $1 \le j \le q$, as follows:

$C' = Q_1 \circ \overleftarrow{Q}_2 \circ Q_3 \circ \overleftarrow{Q}_4 \circ \ldots \circ \overleftarrow{Q}_q$, if $q$ is even and
$C' = Q_1 \circ \overleftarrow{Q}_2 \circ Q_3 \circ \overleftarrow{Q}_4 \circ \ldots \circ Q_q$, if $q$ is odd. □

Since $O_k$ is hamiltonian for $3 \le k \le 13$ [Shields and Savage(2004)], Theorem 9 gives a cycle in $O_{14}$ with at least 75% of the vertices of the graph. Lemmas 7, 8 and Theorem 9 imply that $C'$ has all vertices of $S_k$ and $T_k$. Therefore, to make $C'$ a hamiltonian cycle or path in $O_k$, it remains to add the vertices of $R_k$ to $C'$.

*Modular matchings* were proposed in [Duffus et al.(1994)] for the middle-levels graph $B_k$. Let $A$ be a $k$-subset of $\{1, \ldots, 2k+1\}$. In a matching $m_i$, for $i = 1, \ldots, k+1$, $A$ is adjacent to the set $A \cup \{\overline{a}_j\}$, where

$$j \equiv i + \sum_{a \in A} a \pmod{k+1},$$

and $\overline{a}_j$ is the $j$-th largest element in $\overline{A}$. It was proved that $m_i$ is a perfect matching in $B_k$ [Duffus et al.(1994)] and that $m_i$ can be projected onto the graph $O_k$ by replacing each $(k+1)$-set $A$ by its complement $\overline{A} = \{1, \ldots, 2k+1\} \setminus A$, resulting in either a 2-factor or a perfect matching in $O_k$ [Johnson and Kierstead(2004)]. Notice that $\overline{A}$ is a $k$-set and, therefore, a vertex of $O_k$.

## 3. Main Results

In this section, we show how to obtain a spanning even cactus in $O_k$ from a spanning even cactus in $O_{k-1}$. Since $O_k$ has a hamiltonian cycle for $3 \le k \le 13$, it suffices to prove the statement for $k \ge 14$.

**Definition 10.** *A* peyote *is a spanning even cactus of $O_k$ such that all vertices of $S_k$ and $T_k$ form an even cycle and each vertex of $R_k$ is connected to that cycle by an edge (Figure 4(a)).*

**Lemma 11** ([Mesquita et al.(2014)])**.** *If there exists a hamiltonian cycle $C = (v_1, \ldots, v_q)$ in $O_{k-1}$ such that $|C|$ is even, then $O_k$ has a peyote.*

*Proof (Sketch).* From a hamiltonian cycle $C$ in $O_{k-1}$ such that $|C|$ is even, Theorem 9 constructs a cycle $C'$ with all vertices of $S_k$ and $T_k$ such that $|C'|$ is even. It remains to connect the vertices of $R_k$ to $C'$. By Lemma 6, $|T_k| > |R_k|$. Therefore, a modular matching provides a matching $M$ in the bipartite subgraph $(T_k, R_k)$ that saturates all vertices of $R_k$. The subgraph formed by the cycle $C'$ and the matching $M$ is a peyote of $O_k$. □

**Definition 12** ([Mesquita et al.(2014)])**.** *A* cactoid *is a spanning even cactus of $O_k$ such that all vertices of $S_k$ and $X$, where $X \subseteq T_k$, form an even cycle, all vertices of $T_k \setminus X$ are connected to that cycle by an edge, and each vertex of $R_k$ is connected by an edge to some vertex of $T_k$ (Figure 4(b)).*

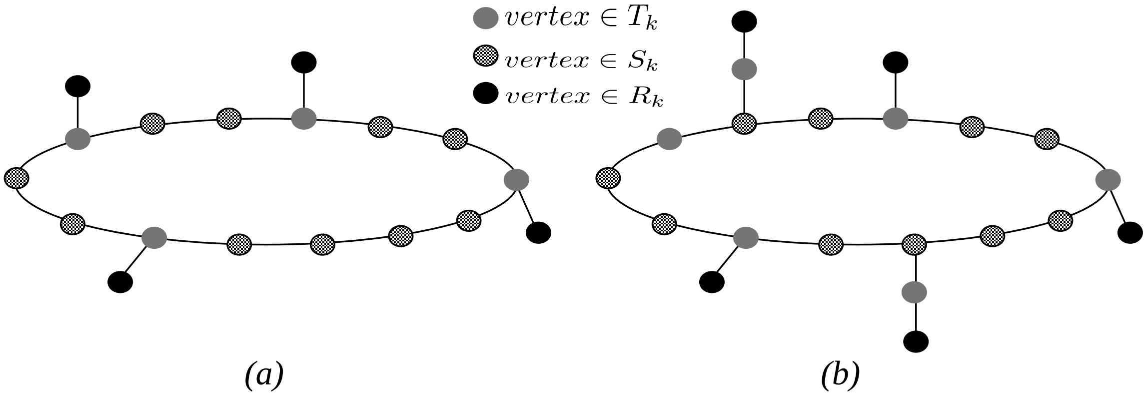



Figure 4: Illustration of (a) a peyote and (b) a cactoid where $X \subsetneq T_k$.

Notice that every peyote is a cactoid where $X = T_k$. Besides, since the even cycle in a cactoid only has vertices of $S_k$ and $T_k$, each vertex of $T_k \setminus X$ is connected to the cycle by an edge to some vertex of $S_k$. Finally, the vertices of $R_k$ are connected to the vertices of $X$ or $T_k \setminus X$. In the last case, instead an edge connected to the even cycle, there is a path of length 2.

**Theorem 13** ([Mesquita et al.(2014)]). *If there exists a cactoid in $O_{k-1}$, then $O_k$ has a cactoid.*

*Proof (Sketch).* Consider that $O_{k-1}$ has a peyote and let $C' = (v_1, v_2, \ldots, v_q)$ be its even cycle. Construct $C'_1$ and $C'_2$ from $C'$ according to Definition 2: $C'_1 = (0v_11, 1v_20, \ldots, 1v_q0)$ and $C'_2 = (1v_10, 0v_21, \ldots, 0v_q1)$. For a vertex $v_i$ in $C'$, $1 \leq i \leq q$, that is adjacent to a vertex $w$ of $R_{k-1}$ in the peyote, add to $C'_1$ and $C'_2$ the edges $\{1w0, 0v_i1\}$ and $\{0w1, 1v_i0\}$. As in Theorem 9, construct $q$ paths by combining the vertices of $C'_1$, $C'_2$ e $T_k$:

(i) If the vertex $v_i$ is not adjacent in $C'$ to a vertex of $R_{k-1}$, then construct a path with three vertices: $(0v_i1, 0\overline{v_i}0, 1v_i0)$;

(ii) If the vertex $v_i$ is adjacent in $C'$ to a vertex $w$ of $R_{k-1}$, then construct a path with five vertices: $(0v_i1, 1w0, 0\overline{w}0, 0w1, 1v_i0)$.

Proceed the concatenation of the $q$ paths as in the proof for Theorem 9, obtaining a cycle $C''$ that has even length. Add the remaining vertices of $T_k$ to $C''$ by joining them to one of its neighbours in $S_k$. By Lemma 6, a modular matching provides a matching $M$ in the bipartite subgraph $(T_k, R_k)$ that saturates all vertices of $R_k$. Therefore, the subgraph formed by the cycle $C''$ and the matching $M$ is a cactoid of $O_k$.

If $O_{k-1}$ has a cactoid that is not a peyote, then the path between $0v_i1$ and $1v_i0$ can be a path with seven vertices: $(0v_i1, 1u0, 0w1, 0\overline{w}0, 1w0, 0u1, 1v_i0)$, where $u \in T_{k-1}$, $w \in R_{k-1}$ and $v_i \in S_{k-1}$. Even so the cycle $C''$ has even length. $\square$

**Corollary 14** ([Mesquita et al.(2014)]). *The prism over the odd graph $O_k$, $k \geq 14$, is hamiltonian.*

*Proof.* Since $O_{13}$ is hamiltonian and $|V(O_{13})|$ is even, by Lemma 11 and Theorem 13, for $k \geq 14$, the odd graph $O_k$ has a cactoid. Therefore, by Proposition 1, $O_k$ is prism-hamiltonian. $\square$

**Theorem 15** ([Mesquita et al.(2014)]). *The odd graph $O_k$, $k \geq 14$, has a cycle with at least $.625|V(O_k)|$ vertices.*

[Horák et al.(2005)] proved that the graph $B_k$ is prism-hamiltonian by determining a spanning 3-connected 3-regular subgraph in $B_k$, since [Paulraja(1993)] proved that graphs with such a spanning subgraph are prism-hamiltonian. We provided an alternative proof for $B_k$, by relating a cactoid in $O_k$ to a spanning even cactus in $B_k$ [Mesquita et al.(2014)].

## 4. Conclusion

In our work, published in [Mesquita et al.(2014)], we proved that there exists a hamiltonian cycle in the prism over each odd graph $O_k$. Previously, it was known that $O_k$ is prism-hamiltonian only for even $k$ [Bueno and Horák(2011)]. Also, we improved the lower bound on the length of the longest cycle of $O_k$ by providing a cycle with at least $.625|V(O_k)|$ vertices. Previously, the best lower bound provided a cycle with less than 3% of the vertices of $O_k$ for $k \geq 10$ [Babai(1979)].

About the contribution of the undergraduate student, our work used Lemma 7 and Theorem 9, two results obtained by the student in a research project from 2011 October to 2012 July, which were presented in SBPO 2013 [Mesquita and Bueno(2013)]. The remaining results were determined by the student in a research project from 2012 August to 2013 July and were presented in the $11^{th}$ Latin American Theoretical INformatics Symposium (LATIN 2014) [Mesquita et al.(2014)]. All results were found by the student under supervision of Letícia R. Bueno and a collaboration with Rodrigo A. Hausen in the proof of Theorem 15.

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## References

**Babai, L.** (1979). Long cycles in vertex-transitive graphs. *Journal of Graph Theory*, 3(3):301–304.

**Biggs, N.** (1979). Some odd graph theory. *Annals of the New York Academy of Sciences*, 319:71–81.

**Bueno, L. R., Faria, L., Figueiredo, C. M. H., and Fonseca, G. D.** (2009). Hamiltonian paths in odd graphs. *Applicable Analysis and Discrete Mathematics*, 3(2):386–394.

**Bueno, L. R. and Horák, P.** (2011). On hamiltonian cycles in the prism over the odd graphs. *Journal of Graph Theory*, 68(3):177–188.

**Čada, R., Kaiser, T., Rosenfeld, M., and Ryjáček, Z.** (2004). Hamiltonian decompositions of prisms over cubic graphs. *Discrete Mathematics*, 286:45–56.

**Duffus, D. A., Kierstead, H. A., and Snevily, H. S.** (1994). An explicit 1-factorization in the middle of the boolean lattice. *Journal of Combinatorial Theory Series A*, 65:334–342.

**Havel, I.** (1983). Semipaths in directed cubes. In Fiedler, M., editor, *Graphs and other Combinatorial Topics*, pages 101–108, Teubner, Leipzig. Teubner-Texte Math.

**Horák, P., Kaiser, T., Rosenfeld, M., and Ryjáček, Z.** (2005). The prism over the middle-levels graph is hamiltonian. *Order*, 22(1):73–81.

**Johnson, J. R.** (2004). Long cycles in the middle two layers of the discrete cube. *Journal of Combinatorial Theory Series A*, 105(2):255–271.

**Johnson, J. R. and Kierstead, H. A.** (2004). Explicit 2-factorisations of the odd graph. *Order*, 21:19–27.

**Kaiser, T., Ryjáček, Z., Král, D., Rosenfeld, M., and Voss, H.-J.** (2007). Hamilton cycles in prisms. *Journal of Graph Theory*, 56:249–269.

**Karp, R. M.** (1972). Reducibility among combinatorial problems. In *Complexity of Computer Computations*, pages 85–103, New York. Plenum Press.

**Lovász, L.** (1970). Problem 11. In *Combinatorial Structures and their Applications*. Gordon and Breach.

**Mesquita, F. C. and Bueno, L. R.** (2013). Ciclos e caminhos hamiltonianos em grafos ímpares. In *XLV Simpósio Brasileiro de Pesquisa Operacional (SBPO), Natal/RN-Brazil*, pages 3006–3013.

**Mesquita, F. C., Bueno, L. R., and Hausen, R. A.** (2014). Odd graphs are prism-hamiltonian and have a long cycle. In *LATIN 2014: Theoretical Informatics*, volume 8392 of *Lecture Notes in Computer Science*, pages 379–390. Springer Berlin Heidelberg.

**Paulraja, P.** (1993). A characterization of hamiltonian prisms. *Journal of Graph Theory*, 17:161–171.

**Savage, C. D. and Winkler, P.** (1995). Monotone gray codes and the middle levels problem. *Journal of Combinatorial Theory Series A*, 70(2):230–248.

**Shields, I. and Savage, C. D.** (2004). A note on hamilton cycles in Kneser graphs. *Bulletin of the Institute for Combinatorics and Its Applications*, 40:13–22.

**Shields, I., Shields, B. J., and Savage, C. D.** (2009). An update on the middle levels problem. *Discrete Mathematics*, 309(17):5271–5277.

**Shimada, M. and Amano, K.** (2011). A note on the middle levels conjecture. *CoRR*, abs/0912.4564.