Algorithms and compact extended formulations for certain simple uncapacitated lot-sizing problems with sales

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ABSTRACT

We consider some simple polynomial uncapacitated lot-sizing problems with sales. We first consider a single level uncapacitated lot-sizing problem with sales and zero demands and show that, differently from the general problem in which demands are considered, the projection of the multicommodity formulation into the original space gives the convex hull of solutions. We then treat the uncapacitated two-level lot-sizing with sales problem for which we present a dynamic programming algorithm and a resulting extended formulation and show how the result can be extended to a generalization of the problem, namely the two-echelon lot-sizing problem in series with intermediate demands and sales.

KEYWORDS. Lot-sizing with sales. Dynamic programming. Extended formulation.

Main Area: Lot-sizing.

1. Introduction

Production planning problems have been studied extensively in the last decades both in a theoretical and in a practical point of view, see Pochet and Wolsey [6] for a survey. Many of the studied problems are extensions of the basic polynomially solvable uncapacitated lot-sizing problem. The presence of potential sales which are limited by some technical or commercial constraints is a characteristic that often appears in industrial settings and this is therefore an interesting area to study.

In a seminal paper in the production planning literature Wagner and Whitin [7] studied the economic lot-size problem and characterized the structure of optimal solutions for the problem in order to devise a polynomial time algorithm.

Loparic et al. [2] studied the uncapacitated lot-sizing with sales problem. The authors give an extended formulation whose projection into the space of original variables describes the convex hull of solutions. In addition, they also describe a family of valid inequalities implied by this projection. Some more general problems with sales were treated for example in Melo and Wolsey [4] and Park [5].

The uncapacitated two-level lot-sizing was studied in Melo and Wolsey [3] and the authors proposed an $O(NT^2 \log NT)$ algorithm from which an extended formulation with $O(NT^3)$ variables and $O(NT^2)$ constraints ($O(NT^3)$ if we consider the nonnegativity constraints) could be obtained using Eppen and Martin's [1] approach. Zhang et al. [8] considered the two-echelon uncapacitated lot-sizing problem in series and presented an $O(NT^4)$ dynamic programming algorithm with a resulting extended formulation with $O(NT^4)$ variables and $O(NT^3)$ constraints ($O(NT^4)$ if we consider the nonnegativity constraints). They also presented a partial description of the convex hull of solutions and used the results in order to treat computationally some extensions of the problem.

In Section 2 we consider the single-level uncapacitated lot-sizing problem with sales and zero demands and show that the projection of the well known multicommodity formulation into the original space gives the convex hull of solutions, which is not the case for the more general problem with nonzero demands. In Section 3 we treat two variants of the two-level lot-sizing with sales that were not yet considered in the literature. In subsection 3.2 we give an $O(NT^4)$ dynamic programming algorithm and a resulting tight extended formulation with $O(NT^4)$ variables and $O(NT^3)$ constraints for the uncapacitated two-level lot-sizing with sales problem. In subsection 3.3 we consider an extension of the two-echelon lot-sizing in which production and sales can happen on both levels and provide an algorithm with the same asymptotic complexity of the special case which can be used to generate a resulting tight extended formulation with the same order of variables and constraints.

2. The single-level uncapacitated lot-sizing with sales and zero demand

In this section we consider a special case of the problem studied in Loparic et al. [2], namely the one in which the demand at each period is equal to zero. The goal is to show that the multicommodity formulation gives the convex hull of the solutions, situation that does not happen for the general problem in which demands are present.

In the single-level uncapacitated lot-sizing with sales and zero demand, there is a single item that can be produced for sale over a discrete horizon of NT time periods. Each sold unit produces a revenue (e_t). A fixed set-up cost (f_t) as well as a per unit production cost (p_t) are implied at each period in case production occurs. At each time period, the amount that can be sold is limited by an upper bound (u_t) and the production is unrestricted (uncapacitated). The goal is to maximize the total revenue minus production cost.

Consider the variables x_t to be the amount produced in period t, y_t to be equal to 1 in case production occurs in period t and 0 otherwise and v_t to be the amount to sell in period t. A standard formulation for the problem is given by $LS0 - STD$.

$$\begin{aligned}
(LS0 - STD) \quad z_D = \max \textstyle\sum_{t=1}^{NT} e_t v_t - \sum_{t=1}^{NT} p_t x_t - \sum_{t=1}^{NT} f_t y_t \quad & (1)\\
\textstyle\sum_{k=1}^{t} x_k \geq \sum_{k=1}^{t} v_k \quad \text{for } 1 \leq t \leq NT, \quad & (2)\\
0 \leq v_t \leq u_t \quad \text{for } 1 \leq t \leq NT, \quad & (3)\\
x_t \leq M y_t \quad \text{for } 1 \leq t \leq NT, \quad & (4)\\
x_t \in \mathbb{R}^1_+ \quad \text{for } 1 \leq t \leq NT, \quad & (5)\\
y_t \in \{0,1\} \quad \text{for } 1 \leq t \leq NT. \quad & (6)
\end{aligned}$$

The objective function maximizes the total revenue minus production cost. Constraints (2) guarantees the production is greater than or equal to the total amount for sale. Constraints (3) limit the amount for sale in each period. Constraints (4) set the fixed set-up variables to 1 in case production occurs. Constraints (5) and (6) are non-negativity and integrality constraints on the variables.

A multi-commodity extended formulation can be obtained by considering the variables w_{kt} to be the amount produced in period k to be sold in period t.

$$\begin{aligned}
(LS0 - MC) \quad z_D = \max \textstyle\sum_{t=1}^{NT} e_t v_t - \sum_{t=1}^{NT} p_t x_t - \sum_{t=1}^{NT} f_t y_t \quad & (7)\\
\textstyle\sum_{k=1}^{t} w_{kt} \geq v_t \quad \text{for } 1 \leq t \leq NT, \quad & (8)\\
0 \leq v_t \leq u_t \quad \text{for } 1 \leq t \leq NT, \quad & (9)\\
w_{kt} \leq u_t y_k \quad \text{for } 1 \leq k \leq t \leq NT, \quad & (10)\\
\textstyle\sum_{t=k}^{NT} w_{kt} = x_k \quad \text{for } 1 \leq k \leq NT, \quad & (11)\\
w_{kt} \in \mathbb{R}^1_+ \quad \text{for } 1 \leq k \leq t \leq NT, \quad & (12)\\
y_t \in \{0,1\} \quad \text{for } 1 \leq t \leq NT. \quad & (13)
\end{aligned}$$

Constraints (8) guarantee the amount to be sold is produced. Constraints (9) limit the amount to be sold in each period. Constraints (10) set the y_k variables to 1 in case production occurs. Constraints (11) link the multi-commodity variables to the original production variables. Constraints (12) and (13) are non-negativity and integrality constraints on the variables.

Observation 1. *When $d_t = 0$, the (t, S, R) inequalities [2] simplify to*

$$\sum_{j \in T \setminus S} x_j + \sum_{j \in S} u^R_{jt} y_j \geq \sum_{j \in R} v_j. \quad (14)$$

Loparic et al. [2] showed that (2)-(3) together with (14) describe $conv(LS0 - STD)$. We now show that for this particular case of the problem with zero demand, the projection of the multi-commodity formulation into the original space of variables gives the convex hull of the feasible solutions.

Proposition 1. $proj_{x,y,v} LS0 - MC = conv(LS0 - STD)$

Proof. It is clear that $conv(LS0 - STD) \subseteq proj_{x,y,v} LS0 - MC$ since $LS0 - MC$ is a valid formulation for the problem. We now want to show that $proj_{x,y,v}(LS0 - MC) \subseteq conv(LS0 - STD)$. Consider a feasible solution $(\hat{w}, \hat{y}, \hat{x}, \hat{v})$ of $LS0 - MC$. We have

$$\sum_{k=1}^{t} \hat{x}_k = \sum_{k=1}^{t} \sum_{j=k}^{t} \hat{w}_{kj} = \sum_{j=1}^{t} \sum_{k=1}^{j} \hat{w}_{kj} \geq \sum_{k=1}^{t} \hat{v}_k. \quad (15)$$

Therefore, (8) implies that (2) is satisfied. Summing (9) over $t \geq k$ for each k, we have that (3) is satisfied. Now, for the (t, S, R) inequalities we have

$$\sum_{j \in T \setminus S} \hat{x}_j + \sum_{j \in S} u^R_{jt} \hat{y}_j = \sum_{j \in T \setminus S} \sum_{k=j}^{NT} \hat{w}_{jk} + \sum_{j \in S} u^R_{jt} \hat{y}_j \geq \tag{16}$$

$$\sum_{j \in T \setminus S} \sum_{\substack{k=j \\ k \in R}}^{t} \hat{w}_{jk} + \sum_{j \in S} \sum_{\substack{k=j \\ k \in R}}^{t} u_k \hat{y}_j \geq \sum_{j \in T \setminus S} \sum_{\substack{k=j \\ k \in R}}^{t} \hat{w}_{jk} + \sum_{j \in S} \sum_{\substack{k=j \\ k \in R}}^{t} \hat{w}_{jk} = \tag{17}$$

$$\sum_{j \in T} \sum_{\substack{k=j \\ k \in R}}^{t} \hat{w}_{jk} = \sum_{k \in R} \hat{v}_k, \tag{18}$$

what implies that the inequalities are also satisfied. □

This implies that constraints (13) can be substituted by

$$y_t \in [0, 1] \quad \text{for } 1 \leq t \leq NT.$$

3. The two-echelon uncapacitated lot-sizing problem in series with sales

In this section we consider the two-echelon uncapacitated lot-sizing problem in series with sales. The goal is to devise extended formulations obtained from dynamic programming algorithms. After introducing the problem we characterize the extreme feasible solutions in subsection 3.1. In subsection 3.2 we treat a special case, namely, the uncapacitated two-level lot-sizing problem with sales for which we present a dynamic programming algorithm and an associated extended formulation obtained using Eppen and Martin's [1] approach. In subsection 3.3 we present a dynamic programming algorithm for the more general two-echelon uncapacitated lot-sizing problem in series with sales.

In the two-echelon uncapacitated lot-sizing problem in series with sales there are two levels of production ($l \in \{0, 1\}$), such that the production at level 1 depends on what was produced at level 0, and a single item with time varying deterministic demand (d^l_t) for both levels over a discrete horizon of NT time periods that have to be satisfied without backlogging. An additional limited amount (up to u^l_t) can be produced at level l for sale in order to get some extra revenue (e^l_t per unit). Production in period t at level l imply a fixed cost f^l_t plus a variable per unit cost p^l_t. We assume that there are no initial and end stocks and that all the data (d, e, f, p) is nonnegative.

Consider the variables x^l_t to be the production quantity at level l in period t, s^l_t the stock quantity at level l at the end of period t, y^l_t equal to 1 if production happens at level l in period t and v^l_t the amount to be sold at level l in period t. A standard formulation for the problem is as follows.

$$(2LS - STD) \qquad \max \sum_{l=0}^{1} \sum_{t=1}^{NT} e^l_t v^l_t - \sum_{l=0}^{1} \sum_{t=1}^{NT} p^l_t x^l_t - \sum_{l=0}^{1} \sum_{t=1}^{NT} f^l_t y^l_t$$

$$s^0_{t-1} + x^0_t = d^0_t + v^0_t + x^1_t + s^0_t \quad \text{for } 1 \leq t \leq NT, \tag{19}$$

$$s^1_{t-1} + x^1_t = d^1_t + v^1_t + s^1_t \quad \text{for } 1 \leq t \leq NT, \tag{20}$$

$$0 \leq v^l_t \leq u^l_t \quad \text{for } 0 \leq l \leq 1, 1 \leq t \leq NT, \tag{21}$$

$$x^l_t \leq M y^l_t \quad \text{for } 0 \leq l \leq 1, 1 \leq t \leq NT, \tag{22}$$

$$x^l_t, s^l_t \in \mathbb{R}^1_+ \quad \text{for } 0 \leq l \leq 1, 1 \leq t \leq NT, \tag{23}$$

$$y^0, y^1 \in \{0, 1\}^{NT}. \tag{24}$$

The objective function maximizes the total revenue minus production cost. Constraints (19) are balance constraints for level 0 and imply that the amount in stock at the beginning of a period plus what was produced in that period at level 0 is equal to the demand in that level plus the amount sold plus the amount produced at level 1 plus what remains as stock at level 0. Constraints (20) are balance constraints for level 1 and the description is similar to the previous one. Constraints (21) restrict the amount of sales for each level and period. Constraints (22) set the setup variables to 1 in case production occurs. Constraints (23) and (24) are respectively nonnegativity and integrality constraints.

3.1. Characterization of extreme feasible solutions

The problem has an associated fixed charge network flow in the form illustrated in Figure 1.

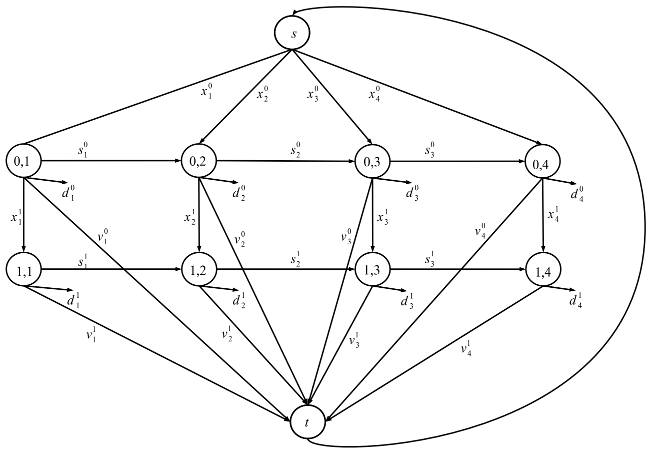

Figure 1: Network representation of the two-echelon uncapacitated lot-sizing problem in series with sales

Observation 2 presents a well known property of fixed charge network flow problems.

Observation 2. *In a basic (or extreme) feasible solution of a fixed charge network flow problem the variables strictly between their upper and lower bounds form an acyclic graph.*

Observation 3 defines regeneration intervals and subintervals for the two-echelon uncapacitated lot-sizing problem without sales.

Observation 3 (Zhang et. al. [8]). *The indices $(1, i_2, 1, j_2)$ form a regeneration interval for $1 \leq i_2 \leq j_2 \leq NT$ if $x^0_1 = d^0_{1i_2} + d^1_{1j_2}$ and $s^0_{i_2} = s^1_{j_2} = 0$ and $s^0_k > 0$ or $d^0_{k+1,i_2} = 0$ for $1 \leq k \leq i_2 - 1$. The indices (i_1, i_2, j_1, j_2) form a regeneration interval for $2 \leq i_1 \leq i_2 \leq j_2 \leq NT$ and $i_1 \leq j_1 \leq j_2$ if $x^0_{i_1} = d^0_{i_1 i_2} + d^1_{j_1 j_2}$ and $s^0_{i_1-1} = s^0_{i_2} = s^1_{j_1-1} = s^1_{j_2} = 0$ and $s^0_k > 0$ or $d^0_{k+1,i_2} = 0$ for $i_1 \leq k \leq i_2 - 1$. In addition, the indices (j_1, j_2) form a regeneration subinterval for level 1 if $x^1_{j_1} = d^1_{j_1 j_2}$ and $s^1_{j_1-1} = s^1_{j_2} = 0$ and $s^1_k > 0$ or $d^1_{k+1,j_2} = 0$ for $j_1 \leq k \leq j_2 - 1$.*

Observation 4. *In an extreme optimal solution every sales variable v^l_k assumes one of its bounds, i.e., either $v^l_k = 0$ or $v^l_k = u^l_k$.*

To verify Observation 4 note that all the variables in the network are uncapacitated with exception of the sales variables, therefore there is a path from the source to node (l, k) formed by basic variables whenever $d^l_k + v^l_k > 0$ for a period k at level l. Assume by contradiction that we have an extreme optimal solution with $0 < v^l_k < u^l_k$ for some l and k. This implies that the arc corresponding to v^l_k forms a cycle with the basic variables on the path from the source to node (l, k), see Figure 2. Therefore the solution is not extreme optimal and we have a contradiction.

Observation 5. *In an extreme optimal solution if production occurs at level l in a period k^l, then the amount produced is used to completely satisfy demand and possible additional sales for an interval of consecutive periods $\{k^l, \ldots, t^l\}$.*

Observation 5 is a direct implication of Observations 2, 3 and 4.

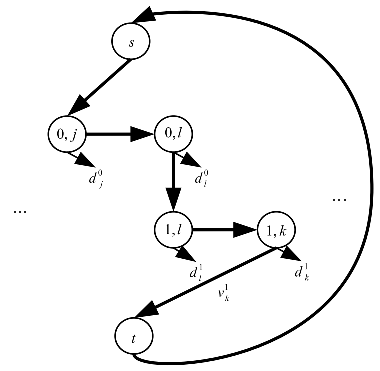

Figure 2: Cycle formed by a v_k^l variable that does not assume one of its bounds

3.2. The special case without intermediate demands

When the intermediate demands and sales are not present ($d_t^0 = 0$ and $v_t^0 = u_t^0 = 0$ for every t), the problem is the sales version of the uncapacitated two-level lot-sizing studied in Melo and Wolsey [3].

3.2.1. A dynamic programming algorithm

In order to generate a dynamic programming recursion, let the following values be defined as:

- $V(i, a, w)$: value of the optional revenue for producing the amount u_w^1 at level 0 in period i and at level 1 in period a, calculated as

$$V(i, a, w) = \max\{0, (e_w^1 - p_i^0 - p_a^1)u_w^1\}. \tag{25}$$

In the solution represented by Figure 3, $V(1, 1, 1)$ and $V(u, j+1, t)$ take nonzero values.

- $B(i, a, b)$: revenue minus cost from satisfying the demands (plus possible additional sales) from periods a to b when the quantity $\sum_{j=a}^{b}(d_j^1 + v_j)$ is produced at level 0 in period i and at level 1 in period a, calculated as

$$B(i, a, b) = \sum_{w=a}^{b} V(i, a, w) - (p_i^0 + p_a^1)d_{ab}^1. \tag{26}$$

In the solution represented by Figure 3, $B(1, 1, j-1)$, $B(u, j, j)$ and $B(u, j+1, t)$ contribute to the revenue minus cost.

- $H(i, j, k)$: maximum revenue minus cost from satisfying the demands (plus possible additional sales) from periods j to k when $\sum_{w=j}^{k}(d_w^1 + v_w)$ units are produced at level 0 in period i, calculated as

$$H(i, j, k) = \max_{j \le w \le k}\{H(i, j, w-1) + B(i, w, k) - f_w^1\}, \tag{27}$$

with $H(i, j, j-1) = 0$. In the solution depicted in Figure 3, we have $H(u, j, t) = H(u, j, j) + B(u, j+1, t) - f_{j+1}^1\}$.

- $G(k)$: optimal revenue for periods 1 to k, calculated as

$$G(k) = \max_{1 \le i \le j \le k}\{G(j-1) + H(i, j, k) - f_i^0\}, \tag{28}$$

with $G(0) = 0$. In the solution represented by Figure 3, $G(t) = G(j-1) + H(u, j, t) - f_u^0$.

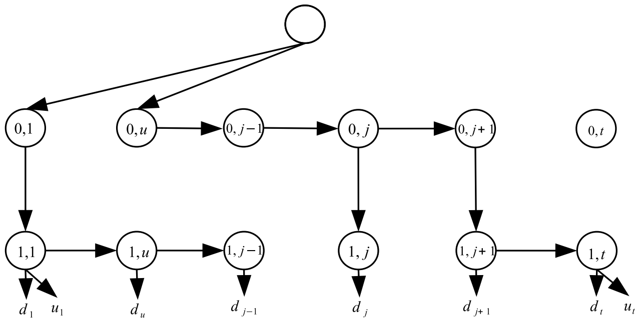

Figure 3: Part of a solution to illustrate the values $G(k)$, $H(i,j,k)$, $B(i,a,b)$ and $V(i,a,w)$.

Proposition 2. *There is an $O(NT^4)$ algorithm for solving the uncapacitated two-level lot-sizing with sales problem.*

Proof. Such an algorithm can be obtained by using a straightforward calculation of the complete set of values involved. All the $V(i,a,w)$ values can be calculated in $O(NT^3)$. All the $B(i,a,b)$ values can be calculated in $O(NT^3)$. The calculation of $H(i,j,k)$ for all i, j and k can be performed in $O(NT^4)$. The values of $G(k)$ for all k can be calculated in $O(NT^3)$. Therefore, the overall running time of the algorithm is $O(NT^4)$. □

3.2.2. An extended formulation

In this section we use the approach of Eppen and Martin [1] to generate a compact extended formulation whose projection into the original space gives the convex hull of solutions.Using the DP recursion given by (25), (26), (27) and (28) we can write the following DP formulation.

$$z_{DPS} = \min G(NT)$$
$$V(i,a,w) \geq (e^1_w - p^0_i - p^1_a)u^1_w \quad \text{for } 1 \leq i \leq a \leq w \leq NT, \tag{29}$$
$$B(i,a,b) \geq \textstyle\sum_{w=a}^{b} V(i,a,w) - (p^0_i + p^1_a)d^1_{ab} \text{ for } 1 \leq i \leq a \leq b \leq NT, \tag{30}$$
$$H(i,j,k) \geq H(i,j,w-1) + B(i,w,k) - f^1_w \text{ for } 1 \leq i \leq j \leq w \leq k \leq NT, \tag{31}$$
$$G(k) \geq G(j-1) + H(i,j,k) - f^0_i \quad \text{for } 1 \leq i \leq j \leq k \leq NT, \tag{32}$$
$$V \in \mathbb{R}^{NT^3}_+, B \in \mathbb{R}^{NT^3}, H \in \mathbb{R}^{NT^3}, G \in \mathbb{R}^{NT}. \tag{33}$$

We present the dual formulation of DP followed by the interpretation of its variables. Associate variables α, β, γ and θ respectively to constraints (29), (30), (31) and (32).

$$z_{DDPS} = \max \textstyle\sum_{i,j,k} \alpha_{ijk}(e^1_k - p^0_i - p^1_j)u^1_k - \sum_{i,j,k} \beta_{ijk}(p^0_i + p^1_j)d^1_{jk} -$$
$$\textstyle\sum_{i,j,w,k} \gamma_{i,j,w,k} f^1_w - \sum_{i,j,k} \theta_{ijk} f^0_l$$
$$\alpha_{iaw} - \textstyle\sum_{j=w}^{NT} \beta_{iaj} \leq 0 \quad \text{for } \leq i \leq a \leq w \leq NT \tag{34}$$
$$\beta_{ijk} - \textstyle\sum_{w=i}^{j} \gamma_{iwjk} = 0 \quad \text{for } 1 \leq i \leq j \leq k \leq NT, \tag{35}$$
$$\textstyle\sum_{w=j}^{k} \gamma_{ijwk} - \sum_{w=k+1}^{NT} \gamma_{ij,k+1,w} - \theta_{ijk} = 0 \text{ for } 1 \leq i \leq j \leq k \leq NT, \tag{36}$$
$$\textstyle\sum_{i=1}^{k} \sum_{j=i}^{k} \theta_{ijk} - \sum_{i=1}^{k+1} \sum_{j=k+1}^{NT} \theta_{i,k+1,j} = 0 \quad \text{for } 1 \leq k \leq NT, \tag{37}$$
$$\textstyle\sum_{i=1}^{NT} \sum_{j=i}^{NT} \theta_{ij,NT} = 1, \tag{38}$$
$$\alpha \in \mathbb{R}^{NT^3}_+, \beta \in \mathbb{R}^{NT^3}_+, \gamma \in \mathbb{R}^{NT^4}_+, \theta \in \mathbb{R}^{NT^3}_+. \tag{39}$$

We use an abuse of notation and denote 'total demand' of an interval $[j,k]$ as $\sum_{i=j}^{k}(d^1_i + v^1_i)$. The variables in the formulation can be interpreted as follows (and are illustrated in Figure 4):

- α_{iaw}: is equal to one (if the variable takes a positive value, then constraint (34) will be tight and therefore this positive value will be 1) if $v^1_w = u^1_w$ with production in period i at level 0 and in period a at level 1 (associated with constraints (29)),
- β_{iab}: is equal to one if the 'total demand' for the interval $[a, b]$ is produced in period i at level 0 and in period a at level 1 (associated with constraints (30)),
- γ_{ijwk}: is equal to one if the 'total demand' for the interval $[w, k]$ is produced in i at level 0 as part of the 'total demand' of an interval starting in period j (associated with constraints (31)),
- θ_{ijk}: is equal to one if the 'total demand' of interval $[j, k]$ is produced in period i at level 0 (associated with constraints (32)).

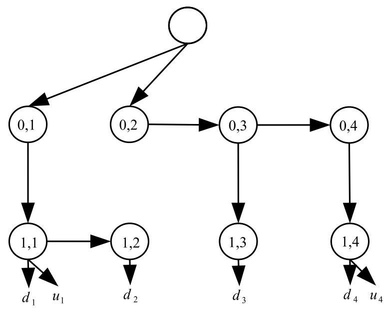

Figure 4: Solution with $\alpha_{111} = \alpha_{244} = \beta_{112} = \beta_{233} = \beta_{244} = \gamma_{1112} = \gamma_{2333} = \gamma_{2344} = \theta_{112} = \theta_{234} = 1$

Constraints (34) link the sales with the production. Constraints (35) state that if a batch $[j, k]$ was produced in period l at level 0 then it is part of a production batch at level 1 starting no later than period j. Constraints (36) indicate that if there is a subbatch $[w, k]$ produced at level one as part of a batch $[j, q]$ at level zero for some w between j and k ($\sum_{w=j}^{k} \gamma_{ijwk} = 1$), then either $k = q$ and $[w, k]$ was the last subbatch of $[j, q]$ ($\theta_{ijk} = 1$), or there is a following subbatch $[k + 1, w]$ of $[j, q]$ ($\sum_{w=k+1}^{NT} \gamma_{i,j,k+1,w} = 1$). Constraints (37) and (38) are shortest path constraints for level zero.

We can link the variables of the formulation DDPS to the original (x, y^0, y^1, v) variables to get the following formulation:

$$(DDPS') \quad \max \sum_{t=1}^{NT} e^1_t v^1_t - \sum_{l=0}^{1} \sum_{t=1}^{NT} p^l_t x^l_t - \sum_{t=1}^{NT} f^0_t y^0_t - \sum_{t=1}^{NT} f^1_t y^1_t$$

$$(34) - (39)$$

$$\sum_{j=t}^{NT} \sum_{k=j}^{NT} \theta_{tjk} \leq y^0_t \quad \text{for } 1 \leq t \leq NT, \tag{40}$$

$$\sum_{j=t}^{NT} \sum_{k=j}^{NT} \beta_{tjk} d^1_{jk} = x^0_t \quad \text{for } 1 \leq t \leq NT, \tag{41}$$

$$\sum_{l=1}^{k} \sum_{j=l}^{k} \sum_{w=k}^{NT} \gamma_{ljkw} \leq y^1_k \quad \text{for } 1 \leq k \leq NT, \tag{42}$$

$$\sum_{j=1}^{k} \sum_{t=k}^{NT} \beta_{jkt} d^1_{kt} = x^1_k \quad \text{for } 1 \leq k \leq NT, \tag{43}$$

$$x^0, x^1 \in \mathbb{R}^{NT}_+, \ v \in \mathbb{R}^{NT}, \ y^0, y^1 \in [0, 1]^{NT}. \tag{44}$$

Let Q^S be the polyhedron described by the constraints (34)-(44). The following theorem is a direct implication of Eppen and Martin's approach.

Theorem 3. *The linear program*

$$\max\{ev - px - fy : (x, y, v, \alpha, \beta, \gamma, \theta) \in Q^S\}$$

solves the two-level problem with sales. Proj$_{x,y,v}(Q^S)$ *is the convex hull of the set of points* (x, y, v) *for which there exists an* s *with* (x, y, v, s) *satisfying (19)-(24).*

3.3. An algorithm for the general problem

We now give a dynamic programing recursion for the general two-echelon lot-sizing problem in series with intermediate demands and sales. Let the following values be defined as:

- $V(i, a, w)$: value of the optional revenue for producing the amount u^1_w at level 0 in period i and at level 1 in period a, calculated as

$$V(i, a, w) = \max\{0, (e^1_w - p^0_i - p^1_a)u^1_w\}. \tag{45}$$

- $B(i, a, b)$: revenue minus cost from satisfying the demands (plus possible additional sales) from periods a to b when the quantity $\sum_{j=a}^{b}(d^1_j + v^1_j)$ is produced at level 0 in period i and at level 1 in period a, calculated as

$$B(i, a, b) = \sum_{w=a}^{b} V(i, a, w) - (p^0_i + p^1_a)d^1_{ab}. \tag{46}$$

- $H(i, j, k)$: maximum revenue minus cost from satisfying the demands (plus possible additional sales) from periods j to k when $\sum_{w=j}^{k}(d^1_w + v^1_w)$ units are produced at level 0 in period i, calculated as

$$H(i, j, k) = \max_{j \le w \le k}\{H(i, j, w - 1) + B(i, w, k) - f^1_w\}, \tag{47}$$

  with $H(i, j, j - 1) = 0$.

- $W(i, b)$: value of the optional revenue for producing the amount u^0_b in period i calculated as

$$W(i, b) = \max\{0, (e^0_b - p^0_i)u^0_b\} \tag{48}$$

- $A(i, k)$: revenue minus cost from satisfying the demands (plus possible additional sales) at level 0 from periods i to k.

$$A(i, k) = \sum_{b=i}^{k} W(i, b) - p^0_i d^0_{ik} \tag{49}$$

- $G(j, k)$: optimal revenue for level 0 until period j and for level 1 until period k, calculated as

$$G(j, k) = \max_{1 \le w \le h \le k}\{G(w - 1, h - 1) + A(w, j) + H(w, h, k) - f^0_w\}, \tag{50}$$

Proposition 4. *There is an* $O(NT^4)$ *algorithm for solving the two-echelon lot-sizing problem in series with intermediate demands and sales.*

Proof. As for the special case considered in Section 3.2, this can be achieved via a straightforward calculation of the different possible values. All the $V(i, a, w)$ can be calculated in $O(NT^3)$. All the $B(i, a, b)$ can be calculated in $O(NT^3)$. All the $H(i, j, k)$ can be calculated in $O(NT^4)$. All the $W(i, b)$ can be calculated in $O(NT^2)$. All the $A(i, k)$ can be calculated in $O(NT^2)$. And all $G(j, k)$ can be calculated in $O(NT^4)$. □

Using the approach of Eppen and Martin [1], as it was done in subsection 3.2.2, an extended formulation with $O(NT^4)$ variables and $O(NT^3)$ constraints (not considering the nonnegativity constraints) can be obtained whose projection into the original space gives the convex hull of solutions.

4. Final Remarks

We considered different uncapacitated production planning problems involving sales and presented polynomial time algorithms and extended formulations. We first showed that for the single-level uncapacitated lot-sizing with sales and zero demand problem the projection of the multicommodity formulation into the original space gives the convex hull of solutions.

We then treated the uncapacitated two-level lot-sizing with sales problem and provided an $O(NT^4)$ algorithm for it. The approach could be extended to the more general two-echelon lot-sizing problem in series with intermediate demands and sales for which an algorithm with the same complexity could be obtained, meaning that it can be solved using an algorithm with the same asymptotic complexity of the variation without sales. Polynomial size extended formulations solving the problem could be obtained using the proposed algorithms. The description of families of strong valid inequalities in the original space of variables is a possible direction for further analysis although we believe that a complete description of the convex hull may be a much more difficult task. It remains an open question whether there are and how to devise polynomial time algorithms for different variants of the problem when capacities on production are considered.

We call the attention to the fact that the results presented here are theoretical and therefore computational experiments should be performed in order to evaluate the practical value of the proposed approaches.

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