SMOOTH DATA ENVELOPMENT ANALYSIS – A NEW MODEL

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ABSTRACT

The present work proposes a different objective function for smooth DEA (Data Envelopment Analysis) models. This new objective function embraces the minimum extrapolation principle, according to which the production set should be as minimal as the constraints allow. The new proposition may be applied to any smooth DEA model that uses a single equation to describe the entire frontier, but we chose a certain model to present the necessary demonstrations. Finally, we present a numerical example that uses a three-dimensional smooth BCC model, in order to compare the new and the traditional objective functions.

KEYWORDS. Smooth DEA, Objective Function, DEA BCC.

Main area: DEA

1. Introduction

Data Envelopment Analysis (DEA) seeks a comparative relation between inputs and outputs of Decision Making Units (DMUs). The method calculates the DMUs´ efficiencies, which, in the original and in the simplest case, is the ratio between the weighted sum of the outputs and the weighted sum of the inputs. These weights are calculated for each DMU so that its efficiency is maximized. Moreover, the efficient DMUs, i.e. the DMUs with the best practices form what is called an efficient frontier.

There are two basic models in DEA: CCR (Charnes *et al.*, 1978) and BCC (Banker *et al.*, 1984). The first assumes constant returns to scale and proportionality between inputs and outputs, while the latter assumes variable returns to scale. This distinction between the two basic models results in different efficient frontiers, which are formed by the efficient DMUs. Therefore, efficiencies in the two models also differ.

The purpose of Smooth DEA models (SOARES DE MELLO *et al.*, 2002, 2004; PEREIRA *et al.*, 2009; NACIF *et al.*, 2009; BRANDÃO & SOARES DE MELLO, 2013; BRANDÃO, 2013) is to correct certain problems of standard DEA models, such as multiple optimal solutions for extreme efficient DMUs. As a positive outcome, these models also eliminate Pareto inefficient regions.

Smooth models are traditionally a Quadratic Problem (QP) seeking the smooth frontier that is, in a certain sense, as close as possible to the original one, and also maintains essential properties from standard DEA. In order to find the most suitable smooth frontier, the traditional objective function minimizes its arc-length (or its multidimensional generalization). Although this is the most adequate objective function for the models that define a different polynomial equation for each segment of the original frontier (SOARES DE MELLO *et al.*, 2001, 2002), it may not lead to the most appropriate result for models that define a single polynomial equation for the entire frontier. This will be further explained in section 3.

Therefore, the present paper proposes a new objective function for smooth models. The new model embraces the minimum extrapolation principle (BANKER *et al*, 1984), according to which the production set should be as minimal as the production assumptions allow. With this new objective function, the smoothing problem is now a Linear Problem (LP), which is simpler to solve. It may be used with any model that defines a single approximation function for the entire frontier and still maintain the same characteristics.

In this paper, we show the implications of the current modifications on the No Optimal Solution Theorem (SOARES DE MELLO *et al.*, 2002). Although the theorem proof differs, its result is the same. In other words, there does not exist the best approximating function for the proposed smoothing-frontier problem.

We also present a three-dimensional version of the smooth BCC model proposed in Brandão (2013), with slight modifications, in order to demonstrate how the new objective function may be applied. Finally, we present a numerical example from Brandão *et al*. (2013) to compare results using the same smooth restrictions with the different objective functions.

2. Bibliographic Review

The Multiplier model, which is one of the dual formulations for classic DEA (Cooper *et al.*, 2000), allows us to calculate trade-offs between inputs and outputs, as well as shadow prices (Coelli *et al.*, 1998). However, we are not able to calculate a unique set of these weights for the extreme-efficient DMUs, which are corners of the efficient frontier, because these DMUs have multiple optimal solutions (SOARES DE MELLO *et al.*, 2002).

To understand why this happens, we should consider the Theorem of Complementary Slacks, which shows that the DMUs’ weights correspond to the coefficients of the hyperplane that is tangent at each point of the efficient frontier (SOARES DE MELLO *et al.*, 2002). Since the original frontier is piecewise linear, it has multiple hyperplanes tangent to each of its corners, where extreme-efficient DMUs are located. Hence, there are multiple optimal sets of weights for each of these DMUs.

SOARES DE MELLO *et al*. (2002) initially proposed Smooth DEA to avoid problems of standard DEA, such as multiple optimal solutions for extreme efficient DMUs. As a positive outcome, these models also eliminate Pareto inefficient regions, where units that are Pareto inefficient are considered to be efficient.

The smooth DEA solution (SOARES DE MELLO *et al*., 2002, 2004; PEREIRA *et al*., 2009; NACIF *et al*., 2009; BRANDÃO & SOARES DE MELLO, 2013, BRANDÃO, 2013) uses Quadratic Programming to replace the original piecewise linear frontier with a new frontier that has derivatives at all points. The new frontier contains all efficient DMUs from standard DEA, it must maintain the essential properties of traditional DEA, and also be as close as possible to the original frontier.

The first model (SOARES DE MELLO *et al*., 2002) proposed a different polynomial equation for each facet of the original frontier. Considering that linear segments compose the original frontier and that a straight line has the minimum arc length between two points, each polynomial equation should have the minimum arc length (or its multi dimensional generalization) in order be as close as possible to the original frontier.

Model (1) represents the Quadratic Problem that smoothens the BCC DEA frontier for the case with one input x and one output y (SOARES DE MELLO *et al*., 2002). In (1), $DMU_i, i = 1, \dots, p$ represents each of the p Pareto efficient DMUs, which are organized in rising order of input values. x_i and y_i are the input and output values for DMU_i, respectively. The frontier will then be described by all the approximating polynomial functions $y = a_i x^2 + b_i x + c_i$, each of them from x_i to x_{i+1}, for $i = 1, \dots, p-1$.

$$
\begin{aligned}
& min \sum_{i=1}^{p-1} \int_{x_i}^{x_{i+1}} \{1 + [(a_i x^2 + b_i x + c)']^2\} \, dx \\
& subject\ to \qquad (1) \\
& a_1 {x_1}^2 + b_1 x_1 + c_1 = y_1 \ (for\ DMU_1) \\
& a_{p-1} {x_p}^2 + b_{p-1} x_p + c_{p-1} = y_p \ (for\ DMU_p) \\
& a_{i-1} {x_i}^2 + b_{i-1} x_i + c_{i-1} = y_i = a_i x^2 + b_i x + c \ (for\ DMU_i, \forall i \in (2, \dots, p-1)) \\
& 2a_{i-1} x_i + b_{i-1} = 2a_i x_i + b_i \ (smooth\ frontier\ for\ DMU_i, \forall i \in (2, \dots, p-1)) \\
& a_i \le 0 \ (frontier\ convexity)
\end{aligned}
$$

The objective function minimizes the sum of the arc lengths of the approximating polynomial functions $y = a_i x^2 + b_i x + c_i$. The first three restrictions ensure that the new frontier contains the extreme efficient DMUs and is continuous; the next restriction ensures continuity of the frontier's derivatives; and the last constraint ensures the BCC property of convexity.

Soares de Mello *et al.* (2004) identified that for cases with more variables, we might not be able to find specific approximating functions for each facet. So they proposed a model with a single polynomial equation for the whole frontier, in the BCC case with 2 inputs (x, y) and 1 output (Z), as in (2). The frontier would be described by a polynomial equation such as $Z = F(x, y) = a + bx + cy + dx^2 + exy + fy^2 + \cdots$

In (2), x_{eff}, y_{eff} are the input values for each extreme efficient DMUs, and Z_{eff} are the output values. The model only considers extreme efficient, which are "corners" of the efficient frontier, is that Soares de Mello *et al.* (2002) detected QP unfeasibility when modeling with more than one non-extreme efficient DMU. Moreover, x_{max}, y_{max} are the greatest inputs of all DMUs.

$$Min\left\{\int_{x_{min}}^{x_{max}}\int_{y_{min}}^{y_{max}}\left[1+\left(\frac{\partial Z}{\partial x}\right)^2+\left(\frac{\partial Z}{\partial y}\right)^2\right]dy\,dx\right\}$$

$$subject\ to \tag{2}$$

$$Z\left(x_{eff},y_{eff}\right)=Z_{eff}\ \forall\ extreme\ efficient\ DMU$$

$$\frac{\partial Z}{\partial x}(x_{max},y_{max})\geq 0$$

$$\frac{\partial Z}{\partial y}(x_{max},y_{max})\geq 0$$

$$\frac{\partial^2 Z}{\partial x^2}\leq 0,\qquad \forall x,y$$

$$\frac{\partial^2 Z}{\partial y^2}\leq 0,\qquad \forall x,y$$

In (2), the first constraint ensures that the smooth frontier includes the same efficient DMUs from standard DEA. The following couple of constraints ensure that the output is an increasing function of the inputs. Finally, the last two constraints would ensure the frontier's convexity. The last two restrictions are not linear, and should be substituted with $d, f \ldots \leq 0$ (SOARES DE MELLO *et al*, 2004) .

Nacif *et al.* (2009) generalized the smooth model for cases with multiple inputs and multiple outputs, using a polynomial for the inputs and another for the outputs. The frontier would be described by their difference as in $U = F(x,y) - H(z,w) = 1 + b_1x + b_2y + b_3x^2 + b_4xy + b_5y^2 + \cdots - c_1z - c_2w - c_3z^2 - c_4zw - c_5w^2 - \cdots$

Further on, Brandão & Soares de Mello (2013) proposed an additional restriction to Smooth BCC models in order to ensure that all DMUs are projected in a viable region of the frontier. In other words, their model guaranteed that no DMU target would have negative inputs.

Brandão (2013) restricted the approximating functions of Smooth BCC models to polynomials with no crossed terms, which are terms with more than one variable. For the case with 2 inputs (x, y) and 1 output (Z), the approximating function would be $Z = F(x,y) = a + bx + cy + dx^2 + ey^2 + fx^3 + gy^3 \ldots$ The author proved that only with this modification, the restrictions in the model ensure the frontier´s convexity.

Brandão *et al.* (2013) used the Smooth Theory to correct the BCC distortion, in which a DMU_0 is necessarily efficient if it is the unique DMU with $x_{i0} = min_{k=1\ldots n}x_{ik}$ or if it is the unique DMU with $y_{j0} = max_{k=1\ldots n}y_{jk}$, where n is the number of DMUs, x_{ik} is the value of DMU k for input i, and y_{jk} is the value of DMU k for output j (ALI, 1993). According to Gomes *et al*. (2012), we may name this distortion efficiency by default, based on the Free Disposal Hull (FDH – DEPRINS *et al*., 1984) approach. Moreover, the authors used a broadened version of this concept, in which every BCC efficient DMU that is not CCR efficient might me considered efficient by default, depending on the smooth frontier results.

Gomes *et al.* (2004) proposed an extension of DEA-ZSG (Zero Sum Game), using the tri-dimensional smooth model of Soares de Mello *et al.* (2004), in order to simplify calculations. Without the smooth frontier, it wouldn't be possible to apply the model to multidimensional cases.

In a different context, other authors proposed continuously differentiable DEA frontiers in order to redistribute resources among the DMUs. In this sense, Avellar *et al.* (2005, 2007) and Silveira *et al.* (2011), proposed hyperbolic, spherical, and parabolic DEA frontiers. Milioni *et al.* (2011) proposed an ellipsoidal frontier model, and showed that their model assures strong efficiency and behaves coherently with sensitivity analysis, which are properties that other technical papers in the found literature do not assure.

3. Modifying the Objective Function

When Soares de Mello *et al.* (2002) first proposed a smooth model, they calculated a different polynomial equation to replace each facet of the original frontier, as explained in the previous section. In this original model, each segment of the smooth frontier needed to be as close as possible to a straight line in order to approximate the new and the original frontiers. This is why the objective function needed to minimize the function's arc length.

However, if we minimize the curve's arc length when calculating a single approximating function for the entire frontier, we will find a frontier that is as close as possible to a single straight line (or its multidimensional generalization) all across the observed DMUs. One may observe that original frontier contrasts significantly with a single straight line.

We must highlight that the smooth model's constraints are typically very strong and do not allow the new frontier to be anywhere near a single straight line. However, it is important to revise the objective function.

First, there is a chance of coming across special cases, and the more we avoid distortions in such cases, the more resilient smooth models will be. Second, authors may wish to modify the original smooth model for certain purposes. Brandao & Soares de Mello (2013), for example, relaxed several equality constraints in order to correct a BCC distortion. Modifications such as the latter may lead to unrealistic results if the objective function is not properly defined. Finally, the new model is a LP, and therefore much simpler to calculate.

Hence, our proposal is to minimize the area limited by the curve, in the two-dimensional case, provided that the constraints are satisfied. In other words, the smooth model will minimize the production possibility region. With this modification, we wish to embrace the minimum extrapolation principle (Banker *et al.*, 1984), according to which the DEA production set should be as minimal as the production assumptions allow.

Figure 1 illustrates the region that the model minimizes, represented by A, for the 1 input (x) and 1 output (y) case. In more general cases, we will minimize the n-dimensional generalization of the area limited by the frontier.

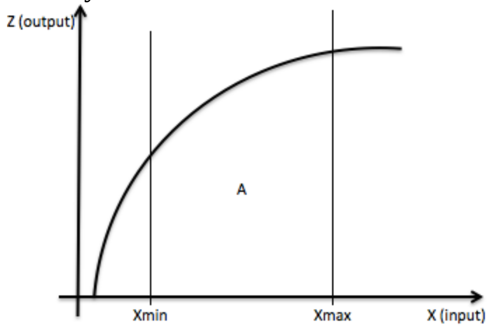

Figure 1 – Minimum production possibility region when the frontier is described by $\boldsymbol{Z} = \boldsymbol{F}(\boldsymbol{x})$

Standard DEA models are in line with this principle because they maximize the DMUs' efficiencies, as long as the restrictions are satisfied. Different models, such as CCR, BCC and FDH, with and without weight restrictions, define different production frontiers, yet they all limit the production frontier as much as the restrictions allow.

We must highlight that the two-dimensional model in Soares de Mello *et al.* (2001, 2002), already embraced the minimum extrapolation principle. This is true because their model defined different polynomial equations for each facet of the frontier. By minimizing their arc lengths, the model approximated the smooth frontier with the original one, as much as the assumptions allowed. Since standard DEA determines the minimum production set, their smooth model also did so.

Therefore, this new objective function is most appropriate for models that calculate a single approximation function for the entire frontier. Yet it may be used with any model with this characteristic because the restrictions are the ones that determine a model's properties, not the objective function. We may observe this from the demonstrations in smooth DEA studies

(SOARES DE MELLO, 2002; NACIF *et al.*, 2009; BRANDÃO, 2013). They all use the restrictions to prove the models' properties.

The degree of the approximating polynomial function, which we select prior to applying the smoothing problem, will also be the same, despite the modification of the objective function. The choice of the degree depends on the polynomial equation (if it includes crossed terms or not) and on the number of efficient DMUs.

Therefore, we are able to apply the new objective function to minimize the production set, using with the restrictions from the three-dimensional smooth BCC model in Soares de Mello *et al.* (2004), the multi-dimensional smooth BCC model in Nacif *et al.* (2009), the smooth CCR model proposed by Pereira *et al.* (2009), etc.

In this paper, we will use a smooth BCC model very similar to the one proposed by Brandão (2013), for the three-dimensional case, with 2 inputs and 1 output. Besides guaranteeing convexity and all existing projections, the model also eliminates the BCC efficiency by default distortion. However, in the present work, we will use the original concept of efficiency by default defined in Ali (1993) and in Gomes *et al*. (2012), instead of its broadened version, defined in Brandão (2013).

In order to illustrate the objective function change, suppose there are two inputs (x, y) and one output (z) and that polynomials of second degree are sufficient for the problem. In this case, the polynomial equation that describes the frontier is $z = F(x, y) = a + bx + cy + dx^2 + ey^2$. In order to minimize the area limited by this frontier, the new objective function should minimize the integral of its function, as in (3).

$$Min\left\{\int_{x_{min}}^{x_{max}}\int_{y_{min}}^{y_{max}} F(x,y)\,dydx\right\} =$$
$$Min\Big\{a(x_{max}-x_{min})(y_{max}-y_{min})+\frac{b}{2}({x_{max}}^2-{x_{min}}^2)(y_{max}-y_{min})+\frac{c}{2}(x_{max}-x_{min})({y_{max}}^2-{y_{min}}^2)+\frac{d}{3}({x_{max}}^3-{x_{min}}^3)(y_{max}-y_{min})+\frac{e}{3}(x_{max}-x_{min})({y_{max}}^3-{y_{min}}^3)\Big\} \tag{3}$$

We may observe that the objective function in (3) is a linear function of a, b, c, d and e, which are the variables for the smoothing problem. Since the model's restrictions are linear (SOARES DE MELLO *et al.*, 2004), the smoothing problem with the new objective function is now a Linear Problem.

3.1 The No Optimal Solution Theorem

The change in the objective function affects the No Optimal Solution Theorem proven in Soares de Mello *et al.* (2002). This Theorem demonstrates that it is impossible to determine a smooth frontier that is closest to the original BCC frontier, when taking into account all possible approximating functions, and using a topology based on the arc length. In other words, it will always be possible to determine a better approximation to the original frontier.

The smoothing problem with the new objective function modifies the theorem proof, but the result remains the same, i.e., it will still be impossible to determine the global optimum approximating function for the smooth frontier.

To demonstrate this, we must use Variational Calculus, which studies the functions that optimize functionals. A functional commonly takes a function in its input argument and returns a scalar, and therefore it is usually considered a "function of functions". Both types of smoothing problems (with the traditional and the new objective functions) are a matter of Variational Calculus. The new objective function, for example, seeks the function that defines the smallest possible area (or its multidimensional generalization).

With a function of two variables, as used in the present work, where $U = f(x, y)$, $f(x_1, y_1) = U_1$, and $f(x_2, y_2) = U_2$, a functional may be represented by:

$$I[U(x,y)] = \iint_{x_1 y_1}^{x_2 y_2} F\left(x, y, U, \frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}\right) dy\, dx$$

To find the function that minimizes the functional, we may use the tri-dimensional form of the Euler-Lagrange equation $\left(F_U - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial U_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial U_y}\right) = 0\right)$ and the adequate boundary conditions (ELSGOTZ, 1980).

In the case presented in this paper, $F = U(x, y)$, so F does not explicitly depend on $\frac{\partial U}{\partial x}$ or $\frac{\partial U}{\partial y}$. In such cases, the Euler-Lagrange equation is reduced to $\frac{\partial F}{\partial U} = 0$. On the other hand, $F = U(x, y)$, so $\frac{\partial F}{\partial U} = 1$, which leads to an impossible problem, i.e., 1=0. In other words, there isn't a global optimum for this smoothing problem, when taking into account all possible approximating functions.

Nevertheless, we may still have good approximating functions to describe the smooth frontier, particularly using polynomial functions (SOARES DE MELLO *et al.*, 2002). Approximating polynomial functions have been used in every smooth DEA paper in the found literature and also in the present work.

3.2 Three-Dimensional Case

For the case with a single output and two inputs, the frontier is described as $Z = F(x, y) = a + b_1 x + b_2 y + c_1 x^2 + c_2 y^2 + d_1 x^3 + d_2 y^3 + \cdots$, with no crossed term in order to guarantee convexity (Brandão, 2013). The objective function that minimizes the possible production frontier, i.e., the region limited by the approximation function, is $Min\left\{\int_{x_{min}}^{x_{max}} \int_{y_{min}}^{y_{max}} F(x, y)\, dy\, dx\right\}$.

We present in (4) the smooth BCC model formulation proposed by Brandão (2013), with the modification aforementioned, and the objective function developed in this paper, for the tri-dimensional case.

$$\begin{aligned}
& Min\left\{\int_{x_{min}}^{x_{max}} \int_{y_{min}}^{y_{max}} F(x, y)\, dy\, dx\right\} \\
& subject\ to \qquad\qquad (4) \\
& F(x_{efi}, y_{efi}) \geq Z_{efi}\ \forall\ DMU\ efficient\ by\ default \\
& F(x_{efi}, y_{efi}) = Z_{efi}\ \forall\ other\ BCC\ efficient\ DMU \\
& \frac{\partial F}{\partial x}(x_{max}, y_{max}) \geq 0 \\
& \frac{\partial F}{\partial y}(x_{max}, y_{max}) \geq 0 \\
& c_1, c_2, d_1, d_2, \ldots \leq 0 \\
& a \leq z_{min}
\end{aligned}$$

The first constraint allows the frontier to be more efficient than the DMUs considered efficient by default. The following constraint ensures that the smooth frontier contains the other efficient DMUs from standard DEA. $\frac{\partial F}{\partial x} \geq 0$ and $\frac{\partial F}{\partial y} \geq 0$ ensure that the output is an increasing function of both inputs. The following restriction imposes convexity to every coefficient of the terms with second derivatives, which is a strong but simple way to guarantee convexity (SOARES DE MELLO *et al*., 2004). Finally, the last constraint guarantees existing targets for all DMUs.

We should mention that these integration limits are an approximation when the output is written as a function of the inputs, but the model is input oriented. In these cases, the range of inputs will be lower then the integration limits. Previous smooth models also use this approximation, with no major distortion. Hence, we will not be concerned about this approximation in this paper.

4. Numerical Example

To illustrate the effects of the objective function modification, we will use the data set presented in Brandão *et al*. (2013) to apply the restrictions of the model presented in (4), but with the traditional and the new objective functions. The original data, the results for standard DEA, and the results for the two objective functions are shown in Table 1.

In Table 1, x represents input 1 (team´s market value), y represents input 2 (FIFA points) and Z represents the output-oriented projection in the frontier (the DMU's target for the MACBETH´s transformation of the tournament ranking). All variables were normalized in Brandão *et al*. (2013), in order to simplify calculations and also avoid eventual distortions.

Problem Data				Standard DEA		Original Obj Function		New Obj Function	
DMUs	x	y	z	Eff BCC	Eff CCR	Z	Eff smooth	Z	Eff smooth
SPAIN	1.00	1.00	1.00	100%	100%	1.000	100.0%	1.000	100.0%
GERMANY	**0.76**	**0.88**	**0.30**	**38%**	**38%**	**0.821**	**36.6%**	**0.813**	**36.9%**
ENGLAND	**0.66**	**0.79**	**0.17**	**25%**	**24%**	**0.719**	**23.6%**	**0.714**	**23.8%**
PORTUGAL	**0.56**	**0.68**	**0.30**	**54%**	**51%**	**0.602**	**49.8%**	**0.600**	**50.0%**
FRANCE	**0.55**	**0.66**	**0.17**	**32%**	**29%**	**0.589**	**28.9%**	**0.587**	**29.0%**
NETHERLANDS	**0.51**	**0.85**	**0.10**	**18%**	**18%**	**0.603**	**16.6%**	**0.602**	**16.6%**
ITALY	0.50	0.67	0.54	100%	100%	0.540	100.0%	0.540	100.0%
RUSSIA	**0.26**	**0.67**	**0.10**	**33%**	**33%**	**0.309**	**32.3%**	**0.314**	**31.9%**
CROATIA	**0.25**	**0.72**	**0.10**	**35%**	**35%**	**0.307**	**32.5%**	**0.312**	**32.1%**
SWEDEN	**0.21**	**0.63**	**0.10**	**42%**	**41%**	**0.238**	**41.9%**	**0.241**	**41.4%**
UKRAINE	**0.18**	**0.39**	**0.10**	**74%**	**50%**	**0.141**	**70.8%**	**0.138**	**72.4%**
C.REPUBLIC	0.17	0.53	0.17	100%	86%	0.170	100.0%	0.170	100.0%
POLAND	**0.15**	**0.36**	**0.10**	**100%**	**58%**	**0.105**	**94.8%**	**0.100**	**100.0%**
DENMARK	0.14	0.70	0.10	56.11%	55.56%	0.191	52.4%	0.192	52.2%
GREECE	0.14	0.65	0.17	100%	100%	0.170	100.0%	0.170	100.0%
IRELAND	**0.11**	**0.62**	**0.10**	**100%**	**71%**	**0.136**	**73.7%**	**0.134**	**74.7%**
AVERAGES				62.97%	55.67%		59.63%		60.07%

Note: Efficient DMUs are in grey and DMUs efficient by default are in bold font.

Table 1 – Comparison between smooth models with different objective functions

With the original objective function, the smooth frontier is described by $Z = F(x, y) = -0.155 + 1.081x + 0.273y - 0.199x^3$. With the new objective function, the polynomial equation is found to be $Z = F(x, y) = -0.193 + 1.216x + 0.341y - 0.318x^2 - 0.046y^2$.

The smooth efficiencies were calculated as in equation (5), which is the same as in standard DEA with one output and output orientation. We would only need to substitute "Z" with "DMU's Output Projection in the Frontier".

$$\%Efficiency_{Smooth} = \frac{DMU's\ Output}{Z} \quad (5)$$

We may observe from Table 1 that most DMUs have higher efficiency values with the new objective function, which results in a higher efficiency average. Besides, Poland, which is considered inefficient in the original smooth model is now considered efficient.

As a sensitivity analysis, we adopted the broad concept of efficiency by default in Brandão *et al.* (2013), i.e., we relaxed the equality restriction for C. Republic, as well as for Poland and Ireland. Using the traditional objective function, the efficiency values for C. Republic and Poland are found to be 97% and 86%, respectively; and the average efficiency is 58.6%. On the other hand, when we apply the new objective function to this relaxed model, the result is the same as the one presented in Table 1, without relaxing the equality restriction for C. Republic. As we may observe from Table 1, both C. Republic and Poland are efficient, and the average efficiency is 60%.

6. Conclusions

The purpose of this work was to propose a new smooth DEA model, by modifying the objective function. The suggested function approximates standard and smooth DEA, because both models calculate the frontier that results in the minimum possible production region, provided that restrictions are satisfied.

The new objective function may be used with any smooth DEA model that calculates a single equation for the entire frontier. This is true because DEA models maximize the DMUs' efficiencies, with the correct constraints, as described in the minimum extrapolation principle. Besides, the models' properties depend on their restrictions, and not on the objective function.

However, in order to demonstrate how to apply the new objective function, we used a variation of the model proposed in Brandão (2013), which eliminates the BCC efficiency by default distortion, and ensures convexity as well as targets in viable regions, for every case. With this modification, we consider efficiency by default only what is traditionally considered as such (ALI, 1993; GOMES *et al*., 2012).

We have also demonstrated the implications of the proposed modification on the No Optimal Solution Theorem (SOARES DE MELLO *et al.*, 2002). Although the theorem proof differed, the result remains the same, so there also does not exist a best approximating function for the proposed smoothing problem.

Finally, we presented a numerical example with two inputs and one output using the data set from Brandão *et al.* (2013). We observed that the average efficiency was higher with the new objective function. Not only that, some DMUs that were efficient in the standard BCC model, but inefficient in the traditional smooth model, became efficient in the new smooth model, presented in this paper.

As a slight sensitivity analysis, we decided to broaden the concept of default efficiency, as in Brandão *et al.* (2013), by relaxing the constraints for all three DMUs that were efficient in the BCC model, but inefficient in the CCR model. With the traditional objective function, the average efficiency was considerably lower, and the smooth frontier didn't contain any of these relaxed DMUs. When we used the same model with the new objective function, two of the three DMUs were considered efficient. Moreover, the frontier equation was found to be exactly the same as in the previous model, with less relaxed restrictions.

This is an indication that the new objective function provides more consistent and reliable results. It may even allow the model to have even more equality restrictions relaxed. If it were possible to relax all equality restrictions, we wouldn't need to use standard DEA prior to smooth models, simply to define which DMUs must be in the frontier. However, this depends on future studies on the subject.

Future studies should also generalize the application of the new objective function, for every case, i.e., the single output and multiple input case, the single input and multiple output case, and the multiple input and output case.

Smooth models with the new objective function are more similar to standard DEA models than the previous smooth models, because the new model embraces the minimum extrapolation principle, as do standard DEA models. The smooth models proposed in this paper are also more robust and resilient because they provide more consistent results, particularly when modifying the equality constraints. Finally, we must point out that the new model is easier to solve because it is a Linear Problem, instead of a Quadratic Problem, as were previous smooth models.

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