A cutting plane algorithm for bounding a strategic pricing problem in electricity markets

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Abstract

We consider a strategic bidding problem under uncertainty in a wholesale energy market, where the economic remuneration of each generator depends on the ability of its own management to submit price and quantity bids. We present a formulation for the problem as a non-convex quadratically constrained quadratic program (QCQP) and propose a cutting plane algorithm to solve an extended linear relaxation for the problem. The linear relaxation is an outer-approximation of the well known SDP relaxation of the QCQP problem. Valid inequalities are added to an initial linear relaxation at each iteration of the cutting plane algorithm, improving the bound computed, including the well known RLT inequalities and sparse SDP cuts that enforce the positive semidefiniteness of the matrix variable. Computational results based on instances derived from the Brazilian system are presented and compare the bounds obtained by different versions of the cutting plane algorithm with the bound obtained with the continuous relaxation of a MILP formulation presented for the problem in the literature.

Keywords: cutting plane algorithm; SDP relaxation; quadratically constrained quadratic problem; strategic pricing.

1 Introduction

In the strategic pricing problem in electricity markets, generators compete for contracts for power sales to distribution companies. They make their price offers for energy production and then are loaded in order of increasing unit price until demand is met. All generators dispatched receive the most expensive unit price charged among them, which corresponds to the marginal cost of short-term or spot price of the system, Hunt(2003).

The problem of determining the optimal price bids for a given company that owns one or more generators is a non-convex problem that may be modeled as a bilevel program, where the leader represents the company that aims to maximize its expected profit, while the follower represents the system operator, that aims to minimize the total cost of the energy production.

Mixed integer linear programming (MILP) reformulations for this strategic bidding problem were proposed in Fampa (2008) and Pereira (2005). The MILP formulation presented in Fampa (2008) was also used in Fampa (2012), in order to obtain the optimal solution of instances derived from the Brazilian system and validate the quality of a genetic algorithm proposed for solving the problem.

In this paper, we consider the reformulation of the strategic pricing bilevel problem in electricity markets as a non-convex quadratically constrained quadratic program (QCQP) and investigate the application of linear programming (LP) in the construction of relaxations for the problem, based on strong positive semidefinite (SDP) relaxations. SDP relaxations of non-convex QCQPs have been studied by a number of researchers, initially inspired by the seminal works of Lovász (1979), Lovász (1991) and Goemans (1995). The research in this field is still very active as shown, for example, on the recent works of Anstreicher (2009), Burer (2008), Fampa (2013), Rendl (2010), Saxena (2010), Saxena (2011) and on the survey paper of Bao (2011).

Although SDP relaxations have been very effective in generating strong bounds for QCQPs, it is well known that the required computation effort to solve the relaxations may be considerable, especially when the size of the relaxation becomes too big due to the inclusion of valid inequalities. To overcome this difficulty, LP outer approximations of the SDP relaxations have been investigated in several works. For example, Margot (2012) investigates LP relaxations of SDP constraints with the aim of capturing most of the strength of SDP relaxations, while still being able to use an LP solver to compute bounds for the problem. In the paper, a basic LP relaxation of the SDP relaxation of the QCQP is strengthened by well known RLT (Reformulation Linearization Technique) inequalities and also by SDP cuts, which are sparse valid cuts based on the spectral decomposition of the matrix variable X that approximate the positive semidefiniteness constraint $X \succeq 0$. In this work we study the application of these LP relaxations for the strategic pricing problem in electricity markets. We also propose a dynamic update of the sparsity of the SDP cut proposed in Margot (2012) and show its effect on the computation for bounds for instances derived from the Brazilian system, that were also addressed in Fampa (2012).

This paper is organized as follows: Section 2 presents mathematical formulations of the strategic bidding problem as a bilevel program and as a QCQP. Section 3 presents the general QCQP and discuss semidefinite relaxations for the problem. Section 4 presents linear relaxations for quadratically constrained quadratic problem, that are outer-approximations of the SDP relaxation. The well know RLT inequalities are presented as well as SDP cuts that approximates the positive semidefinite constraint on the matrix variable. Section 5 presents the algorithm proposed in Margot (2012) to generate sparse SDP cuts. Section 6 presents our cutting plane algorithm to solve the linear relaxation of the strategic bidding problem and Section 7 presents the numerical results comparing the different bounds for the problem. Section 8 concludes the paper.

Notation

In this paper, R^n refers to the n-dimensional Euclidean space, $e_i \in R^n$ represents the i-th unit vector, S^n is the set of $n \times n$ symmetric matrices, S^n_+ is the set of $n \times n$ positive semidefinite symmetric matrices, R^{1+n} and S^{1+n}_+ is used to denote the spaces R^n and S^n_+ with an additional 0-th entry or additional 0-th row and column prefixed. Given two symmetric $n \times n$ matrices X, Y, we let $X \bullet Y = \mathrm{trace}(X^T Y) = \sum_{i,j=1}^n X_{ij} Y_{ij}$ and we use $X \succeq 0$ to denote that the matrix X is positive semidefinite.

2 Strategic Pricing in Electricity Markets

In deregulated electricity markets, generators submit a set of hourly generation prices and available capacities for the following day. Based on these data and on an hourly load forecast, the system operator carries out the following economic dispatch at each time step, Fampa (2008):

$$
\begin{array}{llll}
 & & \text{dual variable} & \\
\text{Minimize}_{g_j} & \sum_{j\in J} \lambda_j g_j, & & \\
\text{subject to} & \sum_{j\in J} g_j = d, & \pi_d & \\
 & g_j \le \bar{g}_j, & \pi_{g_j} & j \in J, \\
 & g_j \ge 0, & & j \in J,
\end{array}
\tag{2.1}
$$

where the input data d, λ_j and $\bar{g}_j$ represent, respectively, load (MWh), price bid $(\$/MWh)$ and generation capacity bid (MWh) of generator j and the variable g_j represents the energy production of generator j (MWh). The optimal value of the dual variable π_d is considered as the system spot price. The profit of each generator $j \in J$, in each time step, corresponds to $(\pi_d - c_j)g_j$, where c_j represents its unit operating cost. Note that c_j may be different from λ_j, its price bid.

The net profit of a generation company E, which may be a utility or an independent power producer that owns several different generation units, is given by:

$$\sum_{j\in E} (\pi_d - c_j) g_j,$$

where E is also used to denote the set of indexes associated to the plants belonging to the company E $(E \subset J)$.

In the optimal price bidding problem, company E aims to determine a set of price bids $\lambda_E = \{\lambda_j, j \in E\}$ that maximize its total net profit, considering the quantity bid of each generator of the company fixed as its maximum generation capacity, denoted by $\bar{g}^*_j$.

The complexity of this problem is increased by the fact that the calculation of π_d and g_j in the dispatch problem (2.1) depends on the knowledge of price vectors for all companies, as well as their generation availability and system load values. However, this information is not available to any single company at the time of its bid. Therefore, the bidding strategy has to take into account the uncertainty around these values. An approach used to deal with the uncertainty on the data of the problem is to define a set of scenarios for the remaining agent's behavior and maximize the profit of the company over all scenarios, in a classical strategic bidding under uncertainty problem. In this case, the bids from generators not belonging to company E and the load are considered uncertain, and represented by a set of scenarios indexed by s, which occur with exogenous probabilities $\{p_s$, s=1,...,S$\}$. The bilevel formulation for the problem is given by

$$
\begin{array}{ll}
\text{Maximize}_{\lambda_E} & \sum_{s\in S} p_s \sum_{j\in E}[\pi_d^s - c_j]g_j^s, \\
\text{subject to} & \\
& \begin{array}{lll}
\text{Minimize}_{g_j^s} & \sum_{s\in S}\sum_{j\in E}\lambda_j g_j^s + \sum_{j\notin E}\lambda_j^{*s} g_j^s, & \\
\text{subject to} & \sum_{j\in J} g_j^s = d^s, & s\in S, \\
& 0 \le g_j^s \le \bar{g}_j^*, \quad j\in E, & s\in S, \\
& 0 \le g_j^s \le \bar{g}_j^{*s}, \quad j\notin E, & s\in S.
\end{array}
\end{array}
\tag{2.2}
$$

The first level of problem (2.2) represents the interest of company E, maximize expected profits), while the second level represents the interest of the system operator (minimize operational costs). The company is classified as leader of the bilevel program and controls the variables λ_j, for $j \in E$, while the system operator is classified as follower and controls the variables g_j^s for $j \in J$, $s \in S$.

Finally replacing the follower linear program by its optimality conditions we derive the following non-convex quadratically constrained quadratic program (QCQP), with a bilinear objective function and one bilinear constraint.

$$
\begin{array}{lll}
\text{Maximize}_{\lambda_j, g_j^s, \pi_d^s, \pi_{g_j}^s} \quad \sum\limits_{s\in S} p_s \sum\limits_{j\in E}[\pi_d^s - c_j]g_j^s & & \\
\text{subject to} & & \\
\sum\limits_{j\in J} g_j^s = d^s, & & s\in S, \\
0 \le g_j^s \le \bar{g}_j^*, & j\in E, & s\in S, \\
0 \le g_j^s \le \bar{g}_j^{*s}, & j\notin E, & s\in S, \\
\pi_d^s - \pi_{g_j}^s - \lambda_j \le 0, & j\in E, & s\in S, \\
\pi_d^s - \pi_{g_j}^s \le \lambda_j^{*s}, & j\notin E, & s\in S, \\
\pi_{g_j}^s \ge 0, & j\in J, & s\in S, \\
\multicolumn{3}{l}{\sum\limits_{s\in S}(\sum\limits_{j\in E}\lambda_j g_j^s + \sum\limits_{j\notin E}\lambda_j^{*s} g_j^s - d^s\pi_d^s + \sum\limits_{j\in E}\bar{g}_j^*\pi_{g_j}^s + \sum\limits_{j\notin E}\bar{g}_j^{*s}\pi_{g_j}^s) = 0.}
\end{array}
\tag{2.3}
$$

3 SDP Relaxations of Quadratically Constrained Quadratic Programs

A general non-convex Quadratically Constrained Quadratic Program (QCQP) may be formulated as:

$$
(\text{QCQP}) \left\{
\begin{array}{lll}
\text{maximize} & x^T Q_0 x + 2q_0^T x + r_0 & \\
\text{subject to} & x^T Q_j x + 2q_j^T x + r_j \le 0, & j = 1,\ldots,m_q \\
& p_j^T x = \theta_j, & j = 1\ldots,m_{l_e} \\
& \alpha_j^T x \le \gamma_j, & j = 1\ldots,m_{l_1} \\
& \beta_j \le \delta_j^T x, & j = 1\ldots,m_{l_2}
\end{array}
\right.
$$

where $Q_j \in S^n$, $q_j \in R^n$, $r_j \in R$, for $j = 0,\ldots,m_q$, $p_j \in R^n$, $\theta_j \in R$, for $j = 1,\ldots,m_{l_e}$, $\alpha_j \in R^n$, $\gamma_j \in R$, for $j = 1,\ldots,m_{l_1}$, $\delta_j \in R^n$, $\beta_j \in R$, for $j = 1,\ldots,m_{l_2}$.

A standard approach to derive a convex relaxation of QCQP is to introduce the variable $Y \in S_+^{1+n}$ in the formulation, obtaining the following lifted reformulation of the problem:

$$
(\text{LIFT}) \begin{cases} \text{maximize} & S_0 \bullet Y \\ \text{subject to} & S_j \bullet Y \leq 0, \qquad j = 1, \ldots, m_q \\ & \quad p_j^T x = \theta_j, \qquad j = 1 \ldots, m_{l_e} \\ & \quad \alpha_j^T x \leq \gamma_j, \qquad j = 1 \ldots, m_{l_1} \\ & \beta_j \leq \delta_j^T x, \qquad j = 1 \ldots, m_{l_2} \\ & Y = \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix}, \quad (X = xx^T) \end{cases}
$$

where $S_j = \begin{pmatrix} r_j & q_j^T \\ q_j & Q_j \end{pmatrix}$, $j = 0, \ldots, m_q$.

The only non-convex constraint in LIFT is the last one, which imposes Y to be a positive semidefinite rank-1 matrix with $Y_{00} = 1$. A convex relaxation of LIFT is then given by the following SDP problem obtained by relaxing the rank-1 constraint:

$$
(\text{SDP}) \begin{cases} \text{maximize} & S_0 \bullet Y \\ \text{subject to} & S_j \bullet Y \leq 0, \qquad j = 1, \ldots, m_q \\ & \quad p_j^T x = \theta_j, \qquad j = 1 \ldots, m_{l_e} \\ & \quad \alpha_j^T x \leq \gamma_j, \qquad j = 1 \ldots, m_{l_1} \\ & \beta_j \leq \delta_j^T x, \qquad j = 1 \ldots, m_{l_2} \\ & S_{m_q+1} \bullet Y = 1, \\ & Y \succeq 0, \qquad (X - xx^T \succeq 0) \end{cases}
$$

where $S_{m_q+1} = e_0 e_0^T$ and $e_0 \in R^{1+n}$.

This approach can be applied to obtain bounds on the optimal value of the QCQP problem using a solver of semidefite programming. However, our aim is to work with linear programming extended relaxations for the problem, which are the subject of the next session.

4 Linear outer-approximation of the SDP relaxation

A linear relaxation of LIFT can be obtained from the previous SDP relaxation, replacing the last constraint, $Y \succeq 0$, by $Y = Y^T$, i.e., imposing only symmetry to the matrix variable Y. We thus get the following linear extended formulation:

$$
(\text{QCQP-L}) \begin{cases} \text{maximize} & S_0 \bullet Y \\ \text{subject to} & S_j \bullet Y \leq 0, \qquad j = 1, \ldots, m_q \\ & \quad p_j^T x = \theta_j, \qquad j = 1 \ldots, m_{l_e} \\ & \quad \alpha_j^T x \leq \gamma_j, \qquad j = 1 \ldots, m_{l_1} \\ & \beta_j \leq \delta_j^T x, \qquad j = 1 \ldots, m_{l_2} \\ & S_{m_q+1} \bullet Y = 1, \\ & Y = Y^T, \end{cases}
$$

QCQP-L is a linear relaxation in x and Y with $n(n+3)/2$ variables and the same number of constraints as SDP. Note that the optimal value of QCQP-L is usually a weak upper bound for the SDP relaxation, as no constraint links the values of x and Y. We propose to strengthen this relaxation by adding with a cutting plane algorithm, RLT inequalities as well as SDP cuts derived from the spectral decomposition of the solution value of the matrix variable Y.

4.1 Adding RLT Inequalities

To strengthen the QCQP-L relaxation, we first consider the well known RLT inequalities (see for example, McCormick (1976), Sherali (1999) and Sherali (1995)). Specifically, the following valid bilinear inequalities

$$\begin{aligned}(x_i - u_i)(x_j - l_j) &\leq 0\\ (x_i - l_i)(x_j - u_j) &\leq 0\\ (x_i - l_i)(x_j - l_j) &\geq 0\\ (u_i - x_i)(u_j - x_j) &\geq 0\end{aligned}$$

generate the RLT inequalities, given by

$$\begin{aligned}X_{ij} - l_j x_i - u_i x_j + l_j u_i &\leq 0\\ X_{ij} - l_i x_j - u_j x_i + l_i u_j &\leq 0\\ X_{ij} - l_j x_i - l_i x_j + l_i l_j &\geq 0\\ X_{ij} - u_j x_i - u_i x_j + u_i u_j &\geq 0.\end{aligned} \tag{4.1}$$

4.2 More RLT Inequalities

Using the idea introduced in Sherali (1995) we multiply the linear constraints among each other and also by each variable of the problem generating valid quadratic constraints to strengthen the identity between X_{ij} and $x_i x_j$ for $i, j = 1, \ldots, n$.

Considering the first type of linear equality constraints in QCQP-L, given by

$$p^T x = \theta \tag{4.2}$$

we derive the valid quadratic equalities

$$(p^T x - \theta)x_i = 0,$$

for each variable x_i in the problem and include in the relaxation

$$p^T x = \theta, \quad \text{and} \quad (p^T x - \theta)x_i = 0, \forall i = 1, \ldots, n. \tag{4.3}$$

Considering now the second type of inequality constraints:

$$\alpha^T x \leq \gamma \tag{4.4}$$

we derive the valid quadratic inequalities

$$(\alpha^T x - \gamma)x_i \leq 0 \quad \forall x_i \geq 0. \tag{4.5}$$

For the third type of linear inequality constraint

$$\delta^T x \geq \beta, \tag{4.6}$$

we derive the valid quadratic inequalities

$$(\delta^T x - \beta)x_i \geq 0 \quad \forall x_i \geq 0. \tag{4.7}$$

Finally, we derive the valid quadratic inequalities derived of :

$$(\alpha^T x - \gamma)(\delta^T x - \beta) \leq 0, \tag{4.8}$$

$$(\alpha^T x - \gamma)(\alpha^T x - \gamma) \geq 0, \tag{4.9}$$

$$(\delta^T x - \beta)(\delta^T x - \beta) \geq 0, \tag{4.10}$$

and include in the relaxation.

Note that all valid quadratic inequalities are introduced in the QCQP-L relaxation as a linear constraint on Y, given by $S \bullet Y \leq 0$, where $S = \begin{pmatrix} r & q^T \\ q & Q \end{pmatrix}$, with properly chosen vectors q and r and symmetric submatrix Q.

4.3 SDP cuts

Finally, we also consider SDP cuts which are based on the fact that the matrix Y is positive semidefinite if and only if

$$v^T Y v \geq 0, \forall v \in R^{n+1} \tag{4.11}$$

The SDP cuts that are considered in our algorithm are detailed in the next section.

5 Generating sparse SDP cuts

We can reformulate the SDP relaxation as the semi-infinite Linear Program:

$$(\text{SDP-L}) \begin{cases} \text{maximize} & S_0 \bullet Y & \\ \text{subject to} & S_j \bullet Y \leq 0, & j = 1, \ldots, m_q \\ & \quad p_j^T x = \theta_j, & j = 1 \ldots, m_{l_e} \\ & \quad \alpha_j^T x \leq \gamma_j, & j = 1 \ldots, m_{l_1} \\ & \beta_j \leq \delta_j^T x, & j = 1 \ldots, m_{l_2} \\ & S_{m_q+1} \bullet Y = 1, & \\ & Y = Y^T, & \\ & v^T Y v \geq 0, \forall v \in R^{n+1}, & \end{cases}$$

Let $\tilde{Y}$ be an arbitrary point in the space of the Y variables. The spectral decomposition of $\tilde{Y}$ is used to decide whether $\tilde{Y}$ is in the SDP cone or not, it is always possible to apply such decomposition to a real symetric matrix, $Y = Y^T$, Golub (2013). Let the eigenvalues and corresponding orthonormal eigenvectors of $\tilde{Y}$ be λ_k and v_k for $k = 1, 2, ..., n$, and assume without loss of generality that $\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n$ and let $t \in 0, ..., n$ such that, $\lambda_t \leq 0 \leq \lambda_{t+1}$. If $t = 0$, then all the eigenvalues are non negative and $\tilde{Y}$ is positive semidefinite.

Otherwise, $v_k^T \tilde{Y} v_k = \lambda_k < 0$ for $k = 1, ..., t$. Hence, the valid cut $v_k^T Y v_k \geq 0$ is violated by $\tilde{Y}$ and may be added to the relaxation to eliminate this solution from the feasible set.

Margot (2012) states that this procedure has two major weaknesses: First, only one cut is obtained from each eigenvector v_k for $k = 1, \ldots, t$, while computing the spectral decomposition requires a non trivial investment in cpu time, and second, the cuts are usually very dense, in other words, almost all entries in $v_k v_k^T$ are nonzero. The authors also state that dense cuts are not good to be used in a cutting plane approach, as they might slow down considerably the reoptimization of the linear relaxation. To address these weaknesses, they propose a procedure to generate sparse cuts from the eigenvectors. The simple idea to compute the sparse cuts is to start with the vector $w = v_k$, for $k = 1, \ldots, t$, and iteratively set to zero some component of w, provided that $w^T \tilde{Y} w$ remains sufficiently negative. If the entries are considered in random order, several cuts can be obtained from a single eigenvector v_k.

The algorithm to generate one sparse cut from a given eigenvector v_k is reproduced from Margot (2012) in the SparseCut procedure, shown in Figure 1. The algorithm receives as input the eigenvector v_k, the matrix $\tilde{Y}$, and two numbers between 0 and 1, pct_{NZ} and pct_{VIOL}, that control the maximum percentage of nonzero entries in the final vector and the minimum violation requested for the corresponding cut, respectively. In the procedure, the parameter $length[v_k]$ identifies the size of vector v_k.

SparseCut(v_k,$\tilde{Y}$, pct_{NZ} and pct_{VIOL})
min_{VIOL}=$(-v_k^T \tilde{Y} v_k)$*pct_{VIOL};
$max_{NZ} = \lfloor length[v_k] * pct_{NZ} \rfloor$;
w=v_k;
$perm$ = random permutation of 1 to $length[w]$;
for $(i = 1, \ldots, length[w])$ **do**
    $z = w$;
    $z[perm[i]] = 0$;
    if $(-z^T \tilde{Y} z > min_{VIOL})$ **then**
        $w = z$;
if *(number of non-zeroes in* $w < max_{NZ}$*)* **then**
    return w;
else
    return $null$;

Fig. 1: Sparsification procedure for SDP cuts

6 Our cutting plane algorithm

In this section we propose a cutting plane algorithm to obtain an upper bound for the strategic pricing problem in energy markets. The idea of the algorithm is to start considering the weak linear extended relaxation of the problem given by formulation (QCQP-L) and then iteratively add the RLT inequalities described in Subsessions 4.1 and 4.2 and the sparce SDP cuts described in Session 5, as proposed in Margot (2012). In our work, however, we propose a dynamic update of the parameters pct_{NZ} and pct_{VIOL} used in the SparceCut routine presented in Session 5. The objective of this dynamic update is to increase the number of SDP cuts that are added to the relaxation on the final steps of the algorithm. We allow the inclusion of less sparse SDP cuts, aiming to improve the bound obtained.

In Figure 2, we present our cutting plane algorithm. The input of the algorithm is composed by the parameters pct_{NZ}, pct_{VIOL}, λ_{MAX} and $MAXCUT$. The two first parameters were introduced in the previous section, the parameter λ_{MAX} is negative and corresponds to the maximum value of the eigenvalue used to generate SDP cuts, and the parameter $MAXCUT$ corresponds to the maximum number of cuts added to the relaxation at each iteration of the cutting plane algorithm. The algorithm is divided into two phases. In the first phase (lines 1-10) the most violated RLT inequalities are iteratively added to the formulation until no RLT inequality is violated by the solution of the current relaxation. Then, on a second phase (lines 11-25), at each iteration of the algorithm, sparse SDP cuts computed by the SparseCut routine, are added. If no sparse eigenvector is generated by the routine with the current values of the parameters pct_{NZ} and pct_{VIOL}, their values are updated (lines 20-22). Each time the parameters are updated their values get 0.5% bigger. In both phases of the algorithm, at most $MAXCUT$ cuts are added to the relaxation at each iteration. The values of the parameters used in our numerical experiments, as well as the stopping criterion adopted in line 12, are presented in the next section.

Input: pct_{NZ}, pct_{VIOL}, λ_{MAX} and $MAXCUT$

1 Let (LR) be the extended linear relaxation (QCQP-L);
2 Let $(\tilde{x}, \tilde{Y})$ be the optimal solution of (LR);
3 Let $nviol$ be the number of RLT inequalities violated by $\tilde{Y}$;
4 **while** $nviol > 0$ **do**
5   **if** $nviol \leq MAXCUT$ **then**
6   Add all the violated RLT inequalities to (LR);
7   **else**
8   Add the $MAXCUT$ most violated RLT inequalities to (LR);
9   Let $(\tilde{x}, \tilde{Y})$ be the optimal solution of (LR);
10   Let $nviol$ be the number of RLT inequalities violated by $\tilde{Y}$;
11 Let λ_k and v_k for $k = 1, \ldots, n$ be respectively, the eigenvalues and corresponding orthogonal eigenvectors of $\tilde{Y}$, such that $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$;
12 **while** *stopping criterion* **do**
13     $tot = 0$; $k = 1$;
14   **while** $\lambda_k < \lambda_{MAX}$ *and* $tot < MAXCUT$ **do**
15         $w_k =$SparseCut$(v_k, \tilde{Y}, pct_{NZ}, pct_{VIOL})$;
16   **if** $(w_k \neq null)$ **then**
17   Add the constraint $w_k^T Y w_k \geq 0$ to (LR);
18             $tot = tot + 1$;
19         $k = k + 1$;
20   **if** $(tot == 0)$ **then**
21         $pct_{NZ} = pct_{NZ} * 1.005$;
22         $pct_{VIOL} = pct_{VIOL} * 1.005$;
23   **else**
24   Let $(\tilde{x}, \tilde{Y})$ be the optimal solution of (LR);
25   Let λ_k and v_k for $k = 1, \ldots, n$ be respectively, the eigenvalues and corresponding orthogonal eigenvectors of $\tilde{Y}$, such that $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$;

Output: The optimal solution value of (LR)

Fig. 2: Our cutting plane algorithm

7 Numerical Results

Our main goal with the numerical experiment discussed, in this section, is to analyze the quality of the upper bounds for the strategic pricing problem in energy markets, computed by our cutting plane algorithm. We present computational results considering some small instances of the problem with configurations derived from the Brazilian power system, as it was done in Fampa (2013). Each instance is characterized by the total number of generators $|J|$, the number of generators that belong to company E, $|E|$, and the number of scenarios $|S|$.

We used the MILP formulation of the problem, presented in Fampa (2008), to obtain the optimal solution of the instances considered, and also the upper bound given by its continuous relaxation.

Our code was implemented in C and compiled with gcc (GNU COMPILE C). All runs were conducted on a 2GB Ram, 2.13GHz Intel Core processor running under Linux Ubuntu, Version: 9.10. The solver CPLEX (IBM(R) ILOG(R) CPLEX(R) Interactive Optimizer 12.5.0.0), Gay (2009), was used to obtain the optimal solution of the instances, with the MILP formulation, the solution of the continuous relaxation of the MILP formulation and the solution of the linear relaxations (LR) in our cutting plane algorithm. LAPACK v. 3.4.2. was used to compute the spectral decomposition of the matrices $\tilde{Y}$.

As the stopping criterion for our cutting plane algorithm, we adopted a time limit of 21600 seconds (6 hours). The stopping criterion in line 12 of Figure 2, is based on the convergence of the algorithm, and was also used in Margot (2012). Considering z_t as the optimal solution of the relaxation at iteration t, the cutting plane algorithm stops if $t \geq 50$ and $z_t \geq (1-0.0001).z_{t-50}$. Note that a purification procedure was implemented to remove the inactive cuts at each t iterations if $t \geq maxIterPurif$ and $z_t \geq (1-0.0001).z_{t-1}$. To avoid premature convergence, the parameter t is reset at each purification. We use $maxIterPur = 5$, updating it in 5% at each iteration, which guarantees a minimum number of iterations to the procedure, i.e., the procedure can only stop if $maxIterPurif \geq 50$, ensuring that $t \geq 50$ may occur.

In our numerical experiments, we have used $pct_{VIOL} = 0.6$, $pct_{NZ} = 0.4$, $\lambda_{MAX} = -0.5$, and $MAXCUT$ equal to 25% of n, the number of variables on our original problem QCQP, i.e., $n = |S|.(2.|J|+1)+|E|$. For example in Table 1, for the first instance, Inst1$_{08,02,02}$, we have $n = 36$ and for the last instance, Inst11$_{10,04,04}$, we have $n = 88$.

Table 1 compares the bounds computed by our cutting plane algorithm (CPA1) with the bounds obtained by our algorithm when we do not update the parameters pct_{VIOL} and pct_{NZ} (CPA2). In this last case the algorithm becomes very similar to what was proposed in Margot (2012). For each instance we ran the cutting plane algorithms 5 times. Because of the random permutation in line 5 of the SparceCut routine presented in Figure 1, we obtain a different result in each run. The results presented in Table 1 correspond to mean values ($\bar{x}$) and standard deviation (σ) considering the 5 runs. In the first column of the table we especify the instance considered.

In order to compare the bounds we first computed for each instance the relative gap between the solution obtained by the cutting plane algorithms (z(CPAi)) and the optimal solution of the problem obtained with the MILP formulation (z(MILP)), which is given by (z(CPAi)-z(MILP))/z(MILP) ×100, for $i = 1, 2$. The mean and standard deviation of these relative gaps are presented in columns 2-3 and 6-7, identified by "MILP". In order to analyze the improvement on the bounds computed by the cutting plane algorithms when compared to the bounds given by the continuous relaxation of the MILP formulation (z(LP)), we also present in the table the mean and standard deviation of the relative gaps given by (z(CPAi)-z(LP))/z(LP) ×100. These statistics are in the columns identified by "LP", i.e., columns 4-5, for CPA1 and 8-9, for CPA2.

We note from the results presented that the cutting plane algorithms generate much stronger bounds than the continuous relaxation of MILP. We also see an improvement on the bounds when using CPA1, compared to CPA2. Concerning the comparison between the running times of algorithms CPA1 and CPA2, we note that in average the time to compute the results with CPA1 is about 34% bigger than with CPA2. The average running time for CPA1 was about 1000 seconds.

Table 1: Comparison of bounds for the strategic bidding problem

	CPA1				CPA2			
	MILP		LP		MILP		LP	
$\text{Inst}_{\|J\|,\|E\|,\|S\|}$	$\bar{x}_{gap}$	σ_{gap}	$\bar{x}_{gap}$	σ_{gap}	$\bar{x}_{gap}$	σ_{gap}	$\bar{x}_{gap}$	σ_{gap}
$\text{Inst1}_{08,02,02}$	0.27%	0.0008	-34.99%	0.0005	4.01%	0.0153	-32.57%	0.0099
$\text{Inst2}_{08,02,03}$	0.15%	0.0005	-32.43%	0.0004	0.20%	0.0014	-32.39%	0.0010
$\text{Inst3}_{08,02,03}$	0.00%	0.0000	-35.05%	0.0000	0.00%	0.0000	-35.05%	0.0000
$\text{Inst4}_{08,02,03}$	0.00%	0.0000	-6.98%	0.0000	0.00%	0.0000	-6.98%	0.0000
$\text{Inst5}_{08,02,03}$	1.27%	0.0023	-24.93%	0.0017	4.75%	0.0070	-22.35%	0.0052
$\text{Inst6}_{08,02,03}$	8.32%	0.0016	-28.38%	0.0011	10.53%	0.0034	-26.91%	0.0023
$\text{Inst7}_{08,02,04}$	3.31%	0,0000	-41.92%	0.0000	3.31%	0.0000	-41.92%	0.0000
$\text{Inst8}_{09,03,02}$	1.28%	0.0007	-48.84%	0.0004	3.44%	0.0035	-47.75%	0.0018
$\text{Inst9}_{09,03,04}$	8.98%	0.0000	-7.16%	0.0000	8.98%	0.0000	-7.16%	0.0000
$\text{Inst10}_{10,04,02}$	4.43%	0.0022	-15.04%	0.0018	4.68%	0.0005	-14.83%	0.0004
$\text{Inst11}_{10,04,04}$	2.42%	0.0000	-33.58%	0.0000	2.42%	0.0000	-33.58%	0.0000

8 Conclusion

In this paper we present a cutting plane algorithm to solve a linear programming relaxation for the strategic bidding problem under uncertainty, which is formulated as a non-convex quadratically constrained quadratic program. In our previous work (Fampa (2013)) we have discussed the application of semidefinite programming relaxations to compute bounds to the strategic bidding problem, considering different relaxations with different strength levels, where we obtain stronger relaxations with the addition of valid inequalities to the weaker ones. As expected from the results presented in the literature for other applications of SDP relaxations, we concluded that we can obtain very tight bounds using strong SDP relaxations of the strategic bidding problem. However, the computational effort to solve the stronger relaxations is quite big. The study motivated the research presented in this work, where we solve an extended linear relaxation of the problem, based on the SDP relaxation, adding at each iteration of the algorithm groups of constraints that strengthen the initial relaxation. These constraints are the well known RLT inequalities and SDP cuts that enforce step by step the positive semidefiniteness of the matrix variable of the original SDP relaxation. Numerical results show that the cutting plane algorithm obtain much better bounds than the continuous relaxation of a MILP formulation of the problem, presented in the literature, and can be computed much faster than the strong SDP relaxations presented in Fampa (2013).

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