Solution of the Problem of Covering Solid Bodies by Spheres using the Hyperbolic Smoothing Technique

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RESUMO

Consideramos o problema da cobertura ótima de corpos sólidos por um determinado número de esferas. A modelagem matemática deste problema conduz a uma formulação *min–max–min* que, além da sua intrínseca natureza multi-nível, tem a significativa característica de ser não-diferenciável. O uso da técnica de Suavização Hiperbólica engendra um simples problema de programação não-linear em um único nível e permite superar as principais dificuldades apresentadas pela formulação original. Para ilustrar o desempenho do método, apresentamos resultados computacionais em um problema de cobertura de um toro. Este é um problema teste cujas soluções ótimas analíticas são sabidas de antemão, pelo menos para um pequeno número de esferas de cobertura.

PALAVRAS CHAVE. Problemas de Localização, Recobrimento, Problemas *min–max–min* , Otimização Não-Diferenciável, Suavização.

Área Principal: Problemas de Recobrimento.

ABSTRACT

We consider the problem of optimally covering solid bodies by a given number of spheres. The mathematical modelling of this problem leads to a *min–max–min* formulation which, in addition to its intrinsic multi-level nature, has the significant characteristic of being non-differentiable. The use of the Hyperbolic Smoothing technique engenders a simple one-level nonlinear programming problem and allows overcoming the main difficulties presented by the original one. To illustrate the performance of the method we present computational results to the problem of covering a torus. This is a problem whose optimal solution is known, at least for a small number of covering spheres.

KEYWORDS. Location Problems, *min–max–min* problems, Non-differentiable programming, Smoothing

Main Area: Covering Problems.

1. Introduction

This paper considers the problem of covering solid bodies by spheres, which has many pratical applications, such as, in radiotherapy treatment of tumors. The mathematical modelling of this problem leads to a *min–max–min* formulation which has the significant characteristic of being strongly non-differentiable.

Rubinov (2006) in his seminar survey about non-smooth optimization classifies this kind of problem as a difficulty one.

There are a very few number of papers considering the solution of covering problems even in spaces with only two dimensions, see Xavier and Oliveira (2005) and Wei et al (2006).

The proposed resolution method adopts an original smoothing strategy which engenders a simple one-level completely differentiable optimization problem. Computational results confirms the adequacy of the proposal.

Let V be a bounded solid body in $\mathbb{R}^3$. The literature defines an *order* 1 *covering* of V by q identical spheres with radius d as a covering in which every point of V must be contained in at least one sphere. In this paper we consider the problem of finding such a covering with the smallest radius d or, in other words, the problem of finding the centers of q spheres that lead to an order 1 covering which has the smallest radius d.

The problem variables are the centers $x_i \in \mathbb{R}^3$, $i = 1, \dots, q$, and the radius d. Consequently the dimension of the solution space is $3q + 1$.

For simplicity the main variable will be defined as

$$x = (x_1, x_2, \dots, x_q, d),$$

such that

$$x \in \mathbb{R}^{3q+1}.$$

For computational purposes the body V is discretized into a finite set of m elementary volumes called voxels: v_j, $j = 1, \dots, m$.

Problems inherent to the covering of $\mathbb{R}^2$ regions by circles, of $\mathbb{R}^3$ regions by spheres, and even regions in higher dimensional spaces have been the object of research for many decades. Important results in the study of these problems appear in Rogers (1964), Toth (1964), Conway (1988) and Hales (1992). The covering of plane domains by a set of circles and ellipses was studied by Galiyev (1995).

This problem has many practical applications such as location of sensors inside large buildings or acoustic sensor networks design Wang and Katabi (2013), Younis and Akkaya (2008).

In medicine this problem can be associated to radiotherapy planning, as presented in the pioneers works by Censor (1988), Michalski et al (2004), Ferris et al (2002) and Ferris et al (2003) . The entire problem is very complex, so Oskoorouchi et al (2011) makes an approach decomposing it into two distinct phases: an initial isocentres determination and a subsequent dose calculation. The 3D covering is much adequate approach for modeling the isocentres determination phase.

The core focus of this paper is the smoothing of the *min–max–min* problem (one of the types discussed by Rubinov (2006)) engendered by the modeling of the covering problem. The smoothing process to be applied is the Hyperbolic Smoothing method, which is an adaptation of the Hyperbolic Penalty method originally introduced by Xavier (1982). This smoothing technique is presented in Santos (1997) for non-differentiable problems in general, and in Chaves (1997) for the *min–max* problem.

By smoothing we fundamentally mean the substitution of an intrinsically non-differentiable three-level problem by a differentiable single-level alternative. This is achieved through the solution of a sequence of differentiable problems which gradually approach the original problem.

This work is organized in the following way. We begin with a detailed introduction to the covering problem in Section 2. The new methodology is described in Section 3. Illustrative computational results are presented in Section 4.

2. The Covering Problem as a Min-Max-Min Problem

In order to formulate the original covering problem as a *min–max–min* problem, we proceed as follows. Let x_i, $i = 1, \ldots, q$ be the centers of the spheres that must cover a region $V \subseteq \mathbb{R}^3$. The set of the centers of the spheres will be represented by $X = (x_1, x_2, \ldots, x_q) \in \mathbb{R}^{3q}$. In order to solve the covering problem, we first discretize the region V into a finite set of m points v_j, $j = 1, \ldots, m$. Given a generic discretization point v_j, we initially calculate the distance from v to the closest sphere center:

$$z_j = \min_{i=1,\ldots,q} \|v_j - x_i\|_2. \tag{1}$$

Distance z_j provides a measurement of the covering for a specific point v_j. The optimal placing of the centers must provide the best quality coverage of the set of discretization points v_j, $j = 1, \ldots, m$ that is, it must minimize the most critical covering. So the covering problem is defined as:

$$\begin{aligned} \text{minimize} \quad & z \\ \text{subject to} \quad & z_j = \min_{i=1,\ldots,q} \|v_j - x_i\|_2, \quad j = 1, \ldots, m \\ & z \geq z_j, \quad j = 1, \ldots, m \end{aligned} \tag{2}$$

Now, let us consider the following relaxation of problem (2):

$$\begin{aligned} \text{minimize} \quad & z \\ \text{subject to} \quad & z_j - \|v_j - x_i\|_2 \leq 0, \quad j = 1, \ldots, m \quad i = 1, \ldots, q \\ & z \geq z_j, \quad j = 1, \ldots, m \end{aligned} \tag{3}$$

This problem is not equivalent to (2) since the variables z_j are not bounded from below, so neither is z. In order to obtain the desired equivalence we must, therefore, modify problem (3). We do so by first letting $\varphi(y)$ denote $max\{0, y\}$ and then observing that from the first set of inequalities in (3), it follows that

$$\sum_{i=1}^{q} \varphi(z_j - \|v_j - x_i\|_2) = 0, \qquad j = 1, \ldots, m. \tag{4}$$

It is possible to grasp the relaxation problem (3) by performing a perturbation $\varepsilon > 0$ of (4):

$$\begin{aligned} \text{minimize} \quad & z \\ \text{subject to} \quad & \sum_{i=1}^{q} \varphi(z_j - \|v_j - x_i\|_2) \geq \varepsilon, \quad j = 1, \ldots, m \\ & z \geq z_j, \quad j = 1, \ldots, m \end{aligned} \tag{5}$$

Since the feasible set of problem (2) is the limit of that of (5) when $\varepsilon \to 0_+$, we can then consider solving (2) by solving a sequence of problems like (5) for a sequence of decreasing values for ε that approaches 0.

The work of Xavier (2005) presents in detail a set of theoretical results associated with the resolution of problem (5). These results ensure the equivalence of problem (5) and problem (2), in terms of the existence of at least one optimal solution (in problem (5), all radii are equal, in other words, $z_j^* = z^*$, $j = 1, \dots, m$). Based on these results, we can greatly reduce the dimension of the problem:

$$\begin{aligned} \text{minimize} \quad & z \\ \text{subject to} \quad & \sum_{i=1}^{q} \varphi(z - \|v_j - x_i\|_2\,) \geq \varepsilon, \quad j = 1, \dots, m \end{aligned} \tag{6}$$

It should be emphasized that problem (6) is defined in a space of dimension $3q+1$, which is much smaller than the space of formulation (5), which has dimension $3q+m+1$. Thus it is more advantageous to computationally solve problem (6) instead of problem (5).

3. Smoothing the Problem

Although problem (6) has reduced dimension, the definition of function φ endows it with an extremely rigid non-differentiable structure, which makes its computational solution very hard. In view of this, the numerical method we adopt for solving problem (6), takes a smoothing approach presented in the work of Xavier and Oliveira (2005). From this perspective, let us define the function:

$$\phi(y, \tau) = \left(y + \sqrt{y^2 + \tau^2}\, \right) / 2 \tag{7}$$

for $y \in \mathbb{R}$ and $\tau > 0$. Function ϕ constitutes an approximation of function φ, since it has the following properties:

(a) $\phi(y,\tau) \; > \; \varphi(y), \quad \forall\, \tau > 0;$

(b) $\lim\limits_{\tau \to 0} \; \phi(y,\tau) \; = \; \varphi(y);$

(c) $\phi(.,\tau)$ is an increasing convex C^∞ function.

These properties allows us to seek a solution to problem (6) by solving a sequence of sub-problems of the form

$$\begin{aligned} \text{minimize} \quad & z \\ \text{subject to} \quad & \sum_{i=1}^{q} \phi(z - \|v_j - x_i\|_2\,, \tau) \geq \varepsilon, \quad j = 1, \dots, m \end{aligned} \tag{8}$$

Just as in other smoothing methods, the solution to the covering problem is obtained by resolving an infinite sequence of constrained minimization sub-problems ($k = 1, 2, \dots$ in the Main Step).

Notice that the algorithm causes τ and ε to approach 0, so the constraints of the sub-problems it solves, given as in (8), tend to those of (6). Also, the algorithm assumes that x^k is a global solution to the kth smoothed sub-problem it solves. Under this hypothesis, and owing to the continuity properties of all functions involved, the sequence $z^1, z^2, \dots$ of optimal values tends to the optimal value of (6).

3.1. The Hyperbolic Penalty Technique

The Hyperbolic Penalty method solves the general nonlinear problem with inequality constraints. It is a very suitable alternative to solving the smoothed problem (8), for it enables a natural coupling with Hyperbolic Smoothing. A brief presentation of the Hyperbolic Penalty method is done in this section. Further reference can be seen in Xavier (1982) and Xavier (2001).

The general nonlinear problem with inequality constraints is represented as follows:

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & g_i(x) \geq 0, \quad i = 1, \cdots, m, \end{array} \tag{9}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ and $g_i : \mathbb{R}^n \to \mathbb{R}, \quad i = 1, \cdots, m.$

The name Hyperbolic Penalty comes from the use of the hyperbolic function (10): that defines a hyperbole with an horizontal asymptote, and another one with an angle $2\,\lambda$, having an intercept ε :

$$P(y, \lambda, \epsilon) = -\lambda y + \sqrt{(\lambda y)^2 + \varepsilon^2} \tag{10}$$

In order to solve (9) by the Hyperbolic Penalty technique, a sequence of intermediate subproblems of the form

$$\text{minimize} \;\; F(x, \lambda^k, \varepsilon^k) = f(x) + \sum_{i=1}^{m} P\left(g_i(x), \lambda^k, \varepsilon^k\right) \tag{11}$$

for $k = 1, 2, \cdots$, are generated.

By this approach, as long as the value y of a generic constraint increases, the value of the penalty decreases asymptotically to zero. As long as this value becomes more negative, i.e. increasing the value of the infeasibility, the penalty value increases asymptotically to the line $-2\,\lambda\,y$. So, it is a penalty that acts coherently in the feasible region, as well in the infeasible one.

The rationality of the Hyperbolic Penalty algorithm is described in Xavier (1982) and quoted below.

The sequence of sub-problems is obtained by the controlled variation of two parameters, λ and ε, in two different phases of the algorithm. Firstly, the parameter λ is increased, implying a significant penalty increment outside the feasible region, and, at the same time, a significant penalty reduction for the feasible region interior points. This process goes on until a feasible point is reached. From there on, λ is fixed, and ε will be decreased sequentially. Thus the interior penalty becomes more irrelevant, keeping the same forbidden level in the exterior region.

3.2. Connecting Hyperbolic Smoothing with Hyperbolic Penalty

The combination of Hyperbolic Smoothing and Hyperbolic Penalty techniques is a very natural and attractive strategy, because both consider the resolution of a sequence of sub-problems. In the smoothing procedure, this sequence is generated by the parameter τ continuously decreasing to zero. In the Hyperbolic Penalty method, at the algorithm second phase, by the parameter ε decreasing to zero.

The connection of Hyperbolic Smoothing and Hyperbolic Penalty consists in the generation of a unique sequence of smooth problems by the simultaneous decreasing of both parameters. This may be achieved by a simple linear coupling of the smoothing parameter τ with the penalty parameter ε. This is used to produce the final algorithm proposed in this paper, presented below in a simplified description.

Algorithm of Hyperbolic Smoothing Covering of Solid bodies by Spheres

1. Choice of initial guess solution x^0 and the initial smoothing and penalty ε^1 and τ^1.

2. Choice of the reduction rate ρ : $0 < \rho < 1$ and the stop tolerance $\delta > 0$. Fix the Hyperbolic Penalty λ : $\lambda = 1$; Do $k = 1$;

3. Do block statement while the stop criterion $|f(x^k) - f(x^{k-1})| < \delta$ is false. Block statement begins here.

4. Solve the differentiable equation (8) with parameter $\tau = \tau^k$, using the Hyperbolic Penalty method penalty parameters $\lambda = 1$ and $\varepsilon = \varepsilon^k$, starting at initial point x^{k-1}, computing solution x^k.

5. Do $\tau^{k+1} = \rho\tau^k$, $\varepsilon^{k+1} = \rho\varepsilon^k$, $k = k + 1$. End block statement (to be repeated while the stop criterion is false).

4. Computational Results

In order to illustrate the functionality of the method, we present some computational results on a test problem associated to a body defined by an accurate mathematical formulation: the regular torus. For this instance, optimal solutions are known beforehand for some particular cases. The starting point X_0 was 10 times randomly chosen and submitted to the algorithm in order to calculate the radius d^*.

The torus was chosen in order to perform a preliminary validation of the method. A torus is a surface of revolution generated by revolving a circle in three-dimensional space about an axis coplanar with the circle. The optimal covering can be easily calculated, as a function of the dimensions of the torus and the number q of spheres, when the number of spheres is small. The torus can be defined parametrically by:

$$x(\theta, \phi) = (R + r\cos\phi)\cos\theta \qquad (12)$$
$$y(\theta, \phi) = (R + r\cos\phi)\sin\theta \qquad (13)$$
$$z(\theta, \phi) = r\sin\phi \qquad (14)$$

where

1. θ and ϕ are angles starting at 0 and ending at 2π, making full circles that start and end at the same point,

2. R is the distance from the center of the tube to the center of the torus,

3. r is the radius of the tube.

A regular ring torus ($r < R$) was chosen for the sphere covering tests, with dimensions $R = 3/4$ and $r = 1/4$ (so that $R + r = 1$ and $R - r = 0.5$). Let γ denote the aspect ratio r/R of the torus (which will always be $1/3$ for the selected torus). The number of spheres q was made to vary from 2 to 40. The optimum solution will be denoted by d^*.

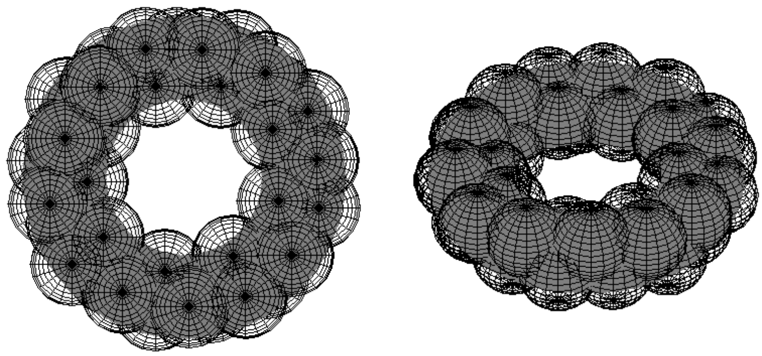
Whenever the number of spheres is small, the optimum solution can be analytically calculated using symmetry: just arrange the spheres uniformly throughout the torus tube, at the same distance ρ^* from the center of the torus. This reasoning leads to the following symmetry constrained solution:

$$\begin{cases} d^* = (R + r)\sin\theta_s & \rho^* = (R + r)\cos\theta_s & \text{if } q < \pi/\arctan(\sqrt{\gamma}) \\ d^* = \sqrt{r^2 + R^2\tan^2\theta_s} & \rho^* = R\sec\theta_s & \text{if } q \geq \pi/\arctan(\sqrt{\gamma}) \end{cases} \qquad (15)$$

where $\theta_s = \pi/q$.

For $\gamma = 1/3$, it was possible to check the optimum solution for q ranging from 2 to 16.

Figure 1 shows graphic depictions of the torus cover with 36 spheres using the Hyperbolic Smoothing technique. This result cannot be analytically verified, but the quality of the solution is clear by visual inspection.



(a) Top view (b) Top slanted view

Figure 1. Covering a Torus using 36 spheres.

Table 1 summarizes the results for these instances. Column q specifies the number of spheres used, column f_{opt} shows the exact analytic solutions d^* (whenever they can be calculated using (15)), column $f_{AHSC-L2}$ shows the computed Hyperbolic Smoothing solutions d^*, and finally columns E_{Mean} and T_{Mean} show the means of deviations of the calculated solutions values and the time spent, in seconds, to find them. These results are a verifiable evidence of the quality of the presented method.

q	f_{ot}	$f_{AHSC-L2}$	E$_{Mean}$	T$_{Mean}$
2	0.100000E01	0.100000E01	0.00	11.21
3	0.866025E00	0.865590E00	0.00	19.03
4	0.707107E00	0.706491E00	0.00	26.94
5	0.587785E00	0.587081E00	0.00	35.56
6	0.500000E00	0.499247E00	0.00	45.61
7	0.439263E00	0.439064E00	0.03	62.34
8	0.398760E00	0.398166E00	0.10	83.77
9	0.370158E00	0.369614E00	0.05	75.08
10	0.349120E00	0.348708E00	0.06	87.26
12	0.320758E00	0.320322E00	0.06	103.40
16	0.291129E00	0.290933E00	0.07	253.29
20	-	0.276674E00	0.49	303.08
24	-	0.269039E00	4.56	538.10
30	-	0.266424E00	4.24	683.48
36	-	0.260362E00	0.66	878.66
40	-	0.252293E00	0.51	979.58

Table 1. Torus with 244,080 voxels

5. Conclusions

In view of the results obtained, where the proposed algorithm performed efficiently and robustly in accordance to the theory developed, we believe that it can be used to solve large, practical optimal covering problems.

Moreover, it must be observed that the methodology introduced in this article can be applied to any *min–max–min* problem. It is well known that an expressive class of global optimization problems can be formulated as *min–max–min* problems. This fact highlights the significance of the employed hyperbolic procedures.

The proposed methodology is not limited to low-dimensional problems. It can easily be extended to be used for coverings of high-dimensional bodies by n-balls (hyperballs).

It is worth noting that the presented results were computed for discretized versions of the intended bodies. That being so, it is not entirely fair to compare the analytical and the computed results in each row of Table 1, because the analytical results were computed considering a really solid torus, not a discretized version of it. Thus, the two values will normally be different, even if it was possible for the presented algorithm to find a full precision solution. Put another way, it is possible that the results achieved by the algorithm are closer to the real values associated with the coverage of the discretized versions of the torus than the calculated analytical values.

An impressive characteristic of the proposed algorithm is its efficiency, which can be easily verified consulting the run times spent to calculate the solutions, as listed in Table 1. The authors were not able to find comparable results in the literature, both in terms of the size of the proposed problems nor in terms of the time spent to find the solutions.

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