A Branch–and–Bound Algorithm for a Special Class of Generalized Multiplicative Programming Programs

Alireza Mohebi Ashtiani
Universidade Tecnológica Federal do Paraná — UTFPR
Departamento Acadêmico de Matemática
Estrada dos Pioneiros, 3131, Jd. Morumbi
86036-370, Londrina – PR
ashtiani@utfpr.edu.br

Paulo Augusto Valente Ferreira
Universidade Estadual de Campinas — UNICAMP
Faculdade de Engenharia Elétrica e de Computação
Av. Albert Einstein, 400, Barão Geraldo
13083-852, Campinas – SP
valente@dt.fee.unicamp.br

ABSTRACT

In this work we propose an outcome space approach for globally solving generalized convex multiplicative problems, a special class of nonconvex problems which involves the minimization of a finite sum of products of convex functions over a nonempty compact convex set. The product of any two or more convex positive functions is not necessarily convex or quasiconvex, and, therefore, the problem may have local optimal solutions that are not global optimal solutions. In the *outcome space*, this problem can be solved efficiently by an algorithm which combines a relaxation technique with the procedure branch–and–bound. Some computational experiences are reported.

KEYWORDS. Global Optimization, Multiplicative Programming, Convex Analysis.

Main Area: Mathematical Programming

RESUMO

Neste trabalho propomos uma abordagem no espaço dos objetivos para resolver globalmente problemas multiplicativos generalizados convexos, uma classe especial de problemas não–convexos que envolve a minimização de uma soma finita de produto de funções convexas sobre um conjunto convexo, compacto e não vazio. O produto de duas ou mais funções convexas positivas necessariamente não é convexa ou quasi–convexa, e, portanto, o problema pode ter soluções ótimas locais que não são soluções ótimas globais. No espaço dos objetivos, este problema pode ser eficientemente resolvido por um algoritmo que combina relaxação com uma técnica de *branch–and–bound*. Algumas experiências computacionais são relatadas.

PALAVRAS CHAVE. Otimização Global, Programação Multiplicativa, Análise Convexa.

Área Principal: Programação Matemática

1. Introduction

Many practical problems in Engineering, Economics and Planning are modeled in a convenient way by Global Optimization problems. The principal objective of this paper is to introduce a new global optimization technique for globally solving a special class of generalized convex multiplicative problems. Convex analysis results allow to reformulate the problem as a semi–infinite problem in an outcome space; a branch and bound algorithm is proposed for solving such problem.

This paper is concerned with the generalized convex multiplicative problem, which consists in minimizing an arbitrary finite sum of products of convex functions over a nonempty compact convex set. The generalized multiplicative programming problem is known as a difficult optimization. Problems of the following forms are considered:

$$\min_{x\in\Omega} \quad f_0(x) + \sum_{i=1}^{p} \prod_{j=1}^{r_i} f_{ij}(x), \tag{1.1}$$

where f_0 and f_{ij} are convex functions defined on $\mathbb{R}^n$. It is also assumed that $\Omega \subset \mathbb{R}^n$ is a nonempty compact convex set and that f_0 and f_{ij} are positive functions over Ω for $i = 1, 2, ..., p$, $j = 1, 2, ..., r_i$. The product of any two or more convex positive functions is not necessarily convex or quasi–convex, and, therefore, problem (1.1) may have local optimal solutions that are not global optimal solutions. In nonconvex global optimization, problem (1.1) has been referred as the generalized convex multiplicative problem. In this paper an approach for globally solving the generalized multiplicative problems 1.1) is proposed and tested.

The conditions $f_0 \equiv 0$ and $p = 1$ characterize the classical convex multiplicative problem. Microeconomics and geometric design are some of the areas where this convex multiplicative programming finds interesting applications. A number of multiplicative programming approaches for solving this problem in the outcome space have been proposed. More recently, a number of branch–and–bound techniques have also been proposed (see Thoai 1991, Kuno 2001 e Oliveira e Ferreira 2008).

The condition $r_i = 2$ for $i = 1, 2, ..., p$ characterizes the classical generalized convex multiplicative problem. Important problems in engineering, financial optimization and economics, among others, rely on mathematical optimization problems of the form (1.1). In (Konno et al., 1994) the problem is projected in the outcome space, where the problem has only m variables, and then solved by an outer approximation algorithm. In (Oliveira and Ferreira, 2010) the problem is projected in the outcome space following the ideas introduced in (Oliveira and Ferreira, 2008), reformulated as an indefinite quadratic problem with infinitely many linear inequality constraints, and then solved by an efficient relaxation–constraint enumeration algorithm. In (Ashtiani and Ferreira, 2011) the authors address the closely related problem of *maximizing* the same objective function, but with f_0 and f_{ij} concave, rather than convex positive functions over Ω. In fact, generalized convex and generalized concave multiplicative problems are found in the fields of quadratic, bilinear and linear zero–one optimization.

In the last decade, many efficient solution algorithms have been proposed for globally solving several particular cases of problem (1.1), especially when $r_i = 2$ or f_0 and f_{ij} are linear (Konno et al. 1994). Problems in which $p > 1$ and $r_i > 2$ have been also addressed, but generally assuming that f_0 and f_{ij} are linear functions (Ryoo and Sahinidis, 2003).

In this paper, a global optimization algorithms based on a suitable reformulation of the problem in the outcome space is proposed. Global minimizers are obtained as the limit of the optimal solutions of a sequence of special programs solved by using a rectangular branch–and–bound procedure.

The paper is organized in five sections, as follows. In Section 2, the problem is reformulated in the outcome space and an outer approximation approach for solving generalized multiplicative problems is outlined. In Sections 3 the relaxation branch–and–bound algorithm is derived.

Some computational experiences with the method described in Section 3 are reported in Section 4. Conclusions are presented in Section 5.

Notation. The set of all n-dimensional real vectors is represented as $\mathbb{R}^n$. The sets of all nonnegative and positive real vectors are denoted as $\mathbb{R}^n_+$ and $\mathbb{R}^n_{++}$, respectively. Inequalities are meant to be componentwise: given $x, y \in \mathbb{R}^n_+$, then $x \geq y$ ($x - y \in \mathbb{R}^n$) implies $x_i \geq y_i, i = 1, 2, ..., n$. Accordingly, $x > y$ ($x - y \in \mathbb{R}^n_{++}$) implies $x_i > y_i, i = 1, 2, ..., n$. The standard inner product in $\mathbb{R}^n$ is denoted as $\langle x, y \rangle$. If $f : \mathbb{R}^n \to \mathbb{R}^m$ is defined on Ω, then $f(\Omega) := \{f(x) \ : \ x \in \Omega\}$. The symbol := means *equal by definition*.

2. The Outcome Space Approach

The outcome space approach for solving problem (1.1) is inspired in a similar approach recently introduced in (Oliveira and Ferreira, 2010) and (Ashtiani and Ferreira, 2011) for solving the classical generalized multiplicative problems ($r_i = 2$ for $i = 1, 2, ..., p$). The objective function in (1.1) can be written as the composition $u(f(x))$, where $u : \mathbb{R}^m \to \mathbb{R}$, $m = pr_i + 1$, is defined by

$$u(y) \ := y_0 + \sum_{i=1}^{p} \prod_{j=1}^{r_i} y_{ij}.$$

The function u can be viewed as a particular aggregating function for the problem of minimizing the vector-valued objective $f := (f_0, f_{11}, ..., f_{1r_1}, ..., f_{p1}, ..., f_{pr_p})$ over Ω (Yu, 1985). The image of Ω under f,

$$\mathcal{Y} \ := f(\Omega), \tag{2.1}$$

is the outcome space associated with problem (1.1). Since f is positive over Ω, it follows that u is strictly increasing over $\mathcal{Y}$ and any optimal solution of (1.1) is Pareto–optimal or efficient (Yu, 1985). It is known from the multiobjective programming literature that if $x \in \Omega$ is an efficient solution of (1.1), then there exists $w \in \mathbb{R}^m_+$ such that x is also an optimal solution of the convex programming problem

$$\min_{x \in \Omega} \langle w, f(x) \rangle. \tag{2.2}$$

Conversely, if $x(w)$ is any optimal solution of (2.2), then $x(w)$ is efficient for (1.1) if $w \in \mathbb{R}^m_{++}$. By defining

$$\mathcal{W} := \Big\{ w \in \mathbb{R}^m_+ \ : \ \sum_{i=1}^{m} w_i = 1 \Big\},$$

the efficient set of (1.1), denoted as $\text{effi}(\Omega)$, can be completely generated by solving (2.2) over $\mathcal{W}$.

The outcome space formulation of problem (1.1) is simply

$$\min_{y \in \mathcal{Y}} \ u(y) \ := y_0 + \sum_{i=1}^{p} \prod_{j=1}^{r_i} y_{ij}. \tag{2.3}$$

The solution approaches which aim at solving problem (1.1) by solving its equivalent problem (2.3) in the outcome space basically differ in the way of representing the (generally) non-convex set $\mathcal{Y}$. The main theoretical result in this paper consists in showing that problem (2.3) admits an equivalent formulation with a convex feasible region. In (Oliveira and Ferreira, 2010) a suitable representation is derived with basis on the following convex analysis result. See (Lasdon, 1970) for a proof.

Lemma 2.1 *Given $y \in \mathbb{R}^m$, the inequality $f(x) \leq y$ has a solution $x \in \Omega$ if and only if y satisfies*

$$\min_{x \in \Omega} \langle w, f(x) - y \rangle \leq 0 \quad \textit{for all} \quad w \in \mathcal{W}.$$

or, equivalently,

$$\max_{x \in \Omega} \langle w, f(x) - y \rangle \geq 0 \quad \textit{for all} \quad w \in \mathcal{W}. \tag{2.4}$$

The main theoretical result of this paper consists in showing that problem (2.3) admits an equivalent formulation with a convex feasible region.

Theorem 2.2 *Let $y^\star$ be an optimal solution of problem*

$$\min_{y \in \mathcal{F}} \; u(y) \; := y_0 + \sum_{i=1}^{p} \prod_{j=1}^{r_i} y_{ij} \tag{2.5}$$

where $\mathcal{F} := \mathcal{Y} + \mathbb{R}^m_+$. Then $y^\star$ is also an optimal solution of (2.3). Conversely, if $y^\star$ solves (2.3), then $y^\star$ also solves (2.13).

Proof. Since for any $x \in \Omega$, $y = f(x)$ is feasible for (2.13), the feasible set of (2.13) contains the feasible set of (2.3). Therefore, the optimal value of (2.13) is a upper bound for the optimal value of (2.3). If $y^\star$ solves (2.13), then

$$\min_{x \in \Omega} \langle w, f(x) - y \rangle \leq 0, \quad \text{for all} \;\; w \in \mathcal{W},$$

and by Lemma 2.1 there exists $x^\star \in \Omega$ such that $f(x^\star) \leq y^\star$. Actually, $f(x^\star) = y^\star$. Otherwise, the feasibility of $f(x^\star)$ for (2.13) and the positivity of u over $\mathcal{F}$ would contradict the optimality of $y^\star$. Since $f(x^\star)$ is feasible for (2.3), we conclude that $y^\star$ also solves (2.3). The converse statement is proved by using similar arguments. □

2.1. Relaxation Procedure

Problem (2.13) has a small number of variables, but infinitely many linear inequality constraints. An adequate approach for solving (2.13) is relaxation. The relaxation algorithm evolves by determining y^k, a global maximizer of u over an outer approximation $\mathcal{F}^k$ of $\mathcal{F}$ described by a subset of the inequality constraints (2.4), and then appending to $\mathcal{F}^k$ only the inequality constraint most violated by y^k. The most violated constraint is found by computing

$$\theta(y) \; := \max_{w \in \mathcal{W}} \; \phi_y(w), \tag{2.6}$$

where

$$\phi_y(w) := \min_{x \in \Omega} \langle w, f(x) - y \rangle. \tag{2.7}$$

Maximin problems as the one described by (2.6) and (2.7) arise frequently in optimization, engineering design, optimal control, microeconomic and game theory, among other areas.

Lemma 2.3 *$y \in \mathbb{R}^m$ satisfies the inequality system (2.4) if and only if $\theta(y) \leq 0$.*

Proof. If $y \in \mathbb{R}^m$ satisfies the inequality system (2.4), then $\min_{x \in \Omega} \langle w, f(x) - y \rangle \leq 0$ for all $w \in \mathcal{W}$, implying that $\theta(y) \leq 0$. Conversely, if $y \in \mathbb{R}^m$ does not satisfy the inequality system (2.4), then $\min_{x \in \Omega} \langle w, f(x) - y \rangle > 0$ for some $w \in \mathcal{W}$, implying that $\theta(y) > 0$. □

If $\theta(y^k) > 0$, then, as a byproduct, the optimal solution of the maximin problem (2.6)-(2.7) characterizes the most violated inequality constraint. As the pointwise minimum of linear functions (indexed by $x \in \Omega$), ϕ_{y^k} is a concave function. Therefore, $\theta(y^k)$ is computed by solving a convex problem.

Some useful properties of θ and ϕ are discussed in Oliveira and Ferreira (2008, 2010). In particular, $f(x(w^0)) - y$ is a subgradient of ϕ_y at any $w^0 \in \mathcal{W}$, and the graph of ϕ_y lies on (or below) the graph of the hyperplane $\phi_y(w^0) + \langle f(x(w^0)) - y, w - w^0\rangle$. This hyperplane is a supporting hyperplane to the hypograph of ϕ_y, which enables piecewise linear approximations for ϕ_y. A l-th approximation for ϕ_y would be

$$\phi_y^l = \min_{1\leq i\leq l} \Big\{\langle w, f(x(w^i)) - y\rangle\Big\}. \tag{2.8}$$

Problem (2.6) is then replaced with the problem of maximizing ϕ_y^l over $\mathcal{W}$, which in turn can be posed as the linear programming problem

$$\left|\begin{array}{ll} \text{maximize} & \sigma \\ \text{subject to} & \sigma \leq \langle w, f(x(w^i)) - y\rangle, \quad i = 1, 2, ..., l, \\ & w \in \mathcal{W}, \ \sigma \in \mathbb{R}. \end{array}\right. \tag{2.9}$$

Let (w^{l+1}, σ^{l+1}) be the optimal solution of the linear program (2.9). If $\sigma^{l+1} - \phi(w^{l+1})$ is less than a prescribed tolerance, then $\theta(y) := \sigma^{l+1}$. Otherwise, a new subgradient $f(x(w^{l+1})) - y)$ is obtained by solving the convex problem in (2.7) and the procedure repeated.

2.2. Basic Algorithm

Consider the initial polytope

$$\mathcal{F}^0 := \Big\{y \in \mathbb{R}^m : 0 < \underline{y} \leq y \leq \overline{y}\Big\}. \tag{2.10}$$

The computations of $\underline{y}$ and $\overline{y}$ demand m convex and m concave minimizations. While the computation of $\underline{y}$ is relatively inexpensive, the computation of $\overline{y}$ requires the solution of m nonconvex problems. However, the usual practice of setting the components of $\overline{y}$ sufficiently large has been successfully applied.

It is readily seen that the minimization of u over $\mathcal{F}^0$ is achieved at $y^0 = \underline{y}$. The utopian point y^0 rarely satisfies the inequality system (2.4), that is, $\theta(y^0) > 0$, in general. By denoting as $w^0 \in \mathcal{W}$ the corresponding maximizer in (2.6), one concludes that y^0 is not in (most violates) the supporting negative half-space

$$\mathcal{H}_+^0 = \Big\{y \in \mathbb{R}^m \ : \ \langle w^0, y\rangle \geq \langle w^0, f(x(w^0))\rangle\Big\}. \tag{2.11}$$

An improved outer approximation for $\mathcal{F}$ is $\mathcal{F}^1 = \mathcal{H}_+^0 \cap \mathcal{F}^0$. If y^1 that minimizes u over $\mathcal{F}^1$ is also such that $\theta(y^1) > 0$, then a new supporting positive half-space $\mathcal{H}_+^1$ is determined, the feasible region of (2.13) is better approximated by $\mathcal{F}^2 = \mathcal{F}^1 \cap \mathcal{H}_+^1$, and the process repeated. At an arbitrary iteration k of the algorithm, the following relaxed program is solved:

$$\min_{y\in\mathcal{F}^k} u(y). \tag{2.12}$$

Problem (2.12) is a linearly constrained problem of the form

$$\left|\begin{array}{ll} \text{minimize} & y_0 + \displaystyle\sum_{i=1}^{p}\prod_{j=1}^{r_i} y_{ij} \\ \text{subject to} & A^{(k)}y \geq b^{(k)}, \\ & \underline{y} \leq y \leq \overline{y}, \end{array}\right. \tag{2.13}$$

where $A^{(k)} \in \mathbb{R}^{k\times m}, b^{(k)} \in \mathbb{R}^k, \underline{y} \in \mathbb{R}^m$ and $\overline{y} \in \mathbb{R}^m$ characterize the matrix representation of $\mathcal{F}^k$.

Thus, the relaxation algorithm for globally solving the generalized multiplicative problem (1.1) assumes the structure below.

Basic Algorithm

Step 0: Find $\mathcal{F}^0$ and set $k := 0$;

Step 1: Solve the generalized multiplicative problem (2.13) using the rectangular branch–and–bound algorithm proposed in the next section, obtaining y^k;

Step 2: Find $\theta(y^k)$ by solving the maximin subproblem (2.6)–(2.7). If $\theta(y^k) < \epsilon$, where $\epsilon > 0$ is a small tolerance, stop: y^k and $x(w^k)$ are ϵ–optimal solutions of (2.3) and (1.1), respectively. Otherwise, define

$$\mathcal{F}^{k+1} := \{y \in \mathcal{F}^k \ : \ \langle w^k, y\rangle \geq \langle w^k, f(x(w^k))\rangle\},$$

set $k := k + 1$ and return to Step 1.

The infinite and finite convergence properties of Algorithm 1 are analogous to those exhibited by the algorithm derived in (Oliveira and Ferreira, 2010) for generalized multiplicative programming.

3. A Rectangular Branch–and–Bound Algorithm

Observe that differently from (2.3), problem (2.13) has a small number of variables, m, but infinitely many linear inequality constraints. An adequate approach for solving (2.13) is to adopt a relaxation technique. The relaxation algorithm evolves by determining y^k, a global minimizer $u(y)$ over an outer approximation $\mathcal{F}^k$ of $\mathcal{F}$, and then appending to $\mathcal{F}^k$ only the inequality constraint most violated by y^k.

3.1. Lower Bound

It is known that a twice–differentiable function $f(y_1, \ldots, y_n) = \beta y_1^{\gamma_1} \ldots y_n^{\gamma_n}$ is convex for $y_j \geq 0,\ \gamma_j \geq 0,\ j = 1, \ldots n$, and $\beta > 0$. Therefore, for underestimating of $f(y)$ we have: if $\gamma_j < 0, \forall j$, then $f(y)$ is already a convex function, while, if $\gamma_j > 0$ for some j, we need to convert $f(y)$ into a new function $f(y, z) = \beta y_1^{\gamma_1} \ldots z_j^{\gamma_j} \ldots y_n^{\gamma_n}$ where $z_j = y_j^{-1}$. Since $f(y, z)$ is already a convex function, we only need to underestimate functions $z_j = y_j^{-1}$ for all j.

Theorem 3.1 (Li et al., 2008) *The lower bound of a nonconvex posynomial function of the form* $f(y_1, \ldots, y_n) = \beta y_1^{\gamma_1} \ldots y_n^{\gamma_n},\ 0 < \underline{y}_j \leq y_j \leq \overline{y}_j,\ \gamma_j \in \mathbb{R}$ *where* $\beta > 0,\ \gamma_j < 0,\ j = 1, \ldots, m,\ \gamma_j > 0,\ j = m + 1, \ldots, n,$ *is obtained by solving the following convex program:*

$$\left|\begin{array}{ll} \text{minimize} & f(y, z) := \beta y_1^{\gamma_1} \ldots y_m^{\gamma_m} z_{m+1}^{-\gamma_{m+1}} \ldots z_n^{-\gamma_n} \\ \text{subject to} & 1 \geq \dfrac{y_j}{\overline{y}_j} + \underline{y}_j z_j - \dfrac{\underline{y}_j}{\overline{y}_j}, \quad j = m + 1, m + 2, \ldots, n. \end{array}\right.$$

Let $\mathcal{R}_y$ denote either the initial rectangle $\mathcal{F}^0 := [\underline{y}, \overline{y}]$, or a subrectangle of it. In each subrectangle, any feasible point of (2.13) provides an upper bound for the optimal value of (2.13). The terms of the form $\prod_{j=1}^{r_i} y_{ij}$ are underestimated by introducing new variable $z \in \mathbb{R}^m$ and an inequality which depends on the bounds on y (Theorem 3.1). A lower bound for the optimal value of (2.13) can be obtained by solving the following convex programming problem:

$$\left|\begin{array}{ll} \text{minimize} & z_0^{-1} + \sum_{i=1}^{p}\prod_{j=1}^{r_i} z_{ij}^{-1} \\ \text{subject to} & A^{(k)}y \geq b^{(k)}, \\ & 1 \geq \frac{y_0}{\overline{y}_0} + \underline{y}_0 z_0 - \frac{\underline{y}_0}{\overline{y}_0}, \\ & 1 \geq \frac{y_{ij}}{\overline{y}_{ij}} + \underline{y}_{ij} z_{ij} - \frac{\underline{y}_{ij}}{\overline{y}_{ij}}, \qquad i = 1, \ldots, p,\ j = 1, \ldots, r_i, \\ & y \in \mathcal{R}_y,\ z \in \mathcal{R}_z, \end{array}\right. \tag{3.1}$$

where $\mathcal{R}_z := \{z \in \mathbb{R}^m \ :\ \frac{1}{\overline{y}_0} \leq z_0 \leq \frac{1}{\underline{y}_0},\ \frac{1}{\overline{y}_{ij}} \leq z_{ij} \leq \frac{1}{\underline{y}_{ij}},\ i = 1, \ldots, p,\ j = 1, \ldots, r_i\}$, $\underline{y}_{ij}$ and $\overline{y}_{ij}(i = 1, \ldots, p,\ j = 1, \ldots, r_i)$ are the bounds on the variables y_{ij} in some subrectangle $\mathcal{R}_y$. Any standard approach from the traditional methods of convex optimization can be applied to solve this problem.

The rectangular branch–and–bound algorithm for globally solving the k-th outer approximation of the generalized multiplicative problem (1.1) assumes the structure below. A similar convergence results for rectangular branch–and–bound algorithms can be found in (Benson, 2002).

Rectangular Branch–and–Bound Algorithm

Step 0: Find $\mathcal{F}^0$, let some accuracy tolerance $\epsilon BB > 0$ and the iteration counter $k = 0$.

Step 1: Define the initial list $\mathcal{L}_0 := \{\mathcal{F}^0\}$, and let L_0 and U_0 be a lower and an upper bound for the optimal value of problem (3.1), with $\mathcal{R}_y = \mathcal{F}^0$.

Step 2: While $U_k - L_k > \epsilon BB$,

(i) Choose $\mathcal{R}_y \in \mathcal{L}_k$ such that the such that the lower bound over $\mathcal{R}_y$ is equal to L_k;

(ii) Split $\mathcal{R}_y$ along one of its longest edges into $\mathcal{R}_{y_I}$ and $\mathcal{R}_{y_{II}}$;

(iii) Define

$$\mathcal{L}_{k+1} := \left(\mathcal{L}_k - \{\mathcal{R}_y\}\right) \bigcup \left\{\mathcal{R}_y^I, \mathcal{R}_y^{II}\right\};$$

(iv) Compute lower and upper bounds for the optimal values of problems (3.1) with $\mathcal{R}_y = \mathcal{R}_y^I$ and (3.1) with $\mathcal{R}_y = \mathcal{R}_y^{II}$, set L_{k+1} and U_{k+1} as the minima lower and upper bounds over all subrectangles $\mathcal{R}_y \in \mathcal{L}_{k+1}$, and $k := k + 1$.

4. Computational Experiments

The basic algorithm and the retangular branch–and–bound algorithm, which solve outer approximations of generalized multiplicative problems were coded in `MATLAB` (V. 7.0.1)/Optimization Toolbox (V. 4) and run on a personal Pentium IV system, 2.00 GHz, 2048MB RAM. The tolerances for the ϵ–convergences of algorithm was fixed at 10^{-5} while the tolerance for the convergence of the branch–and–bound algorithm was fixed at 0.05. In order to illustrate the convergence of the global optimization algorithms proposed, the following example has been considered.

Example 4.1 *Consider the illustrative problem discussed in (Schaible and Sodini, 1995) where an alternative algorithm has been proposed:*

$$\left|\begin{array}{ll} \text{minimize} & (x_1+1)+(2x_1-3x_2+13)(x_1+x_2-1) \\ & \\ \text{subject to} & -x_1+2x_2 \leq 8, \\ & -x_2 \leq -3, \\ & x_1+2x_2 \leq 12, \\ & x_1-2x_2 \leq -5, \\ & x_1 \geq 0, \quad x_2 \geq 0. \end{array}\right.$$

The functions f_0, f_{11} and f_{12} ($p = 1$ and $r_1 = 2$) are convex and positive over the feasible convex, compact and nonempty region Ω. The lower and upper bounds on $y = (y_0, y_{11}, y_{12})$ are $\underline{y} = (1.00, 1.00, 2.00)$ and $\overline{y} = (4.50, 7.25, 6.75)$, respectively. The relaxation algorithm converges in 2 iterations to $x^\star = (0.00, 4.00)$, the same solution found in (Schaible and Sodini, 1995), providing the ϵBB—optimal value $u^\star = 4.00$. At the convergence, $UB_2 - LB_2 = 0.0274$.

Example 4.2 *As a second numerical experience, consider the generalized convex problem obtained from (Konno et al., 1994) and (Oliveira and Ferreira, 2010) where outer approximation algorithms are proposed for globally solving a special class of (1.1). The problem is*

$$\left|\begin{array}{ll} \text{minimize} & (3x_1-4x_2+15)+(x_1+2x_2-1.50)(2x_1-x_2+4) \\ & +(x_1-2x_2+8.50)(2x_1+x_2-1) \\ \text{subject to} & 5x_1-8x_2 \geq -24, \\ & 5x_1+8x_2 \leq 44, \\ & 6x_1-3x_2 \leq 15, \\ & 4x_1+5x_2 \geq 10, \\ & x_1 \geq 0. \end{array}\right.$$

Again, the functions $f_0, f_{11}, f_{12}, f_{21}$ and f_{22} ($p = 2$ and $r_1 = r_2 = 2$) are convex and positive over the feasible convex, compact and nonempty region Ω. The lower and upper bounds on $y = (y_0, y_{11}, y_{12}, y_{21}, y_{22})$ are $\underline{y} = (3.00, 1.00, 1.00, 2.00, 2.00)$ and $\overline{y} = (22.50, 9.00, 9.00, 11.00, 11.00)$, respectively (Konno et al., 1994). The relaxation algorithm converges in 3 iterations to $x^\star = (0.00, 3.00)$, the same solutions found in (Konno et al., 1994) and (Oliveira and Ferreira, 2010), providing the ϵBB—optimal value $u^\star = 12.50$. At the convergence, $UB_3 - LB_3 = 0.0312$.

The proposed algorithm converged in 0.83 and 2.11 seconds solving Examples 4.1 and 4.2, respectively. The CPU time of the proposed algorithm tends to increase rapidly as the number of product terms, p, increase, because the computational effort demanded by the branch–and–bound algorithm grows exponentially with p.

5. Conclusions

In this work we proposed a global optimization approach for generalized convex multiplicative programs. By using convex analysis results the problem was reformulated in the outcome space as an optimization problem with infinitely many linear inequality constraints, and then solved through a relaxation branch–and–bound algorithm. Experimental results have attested the viability and efficiency of the proposed global optimization algorithm, which is, in addition, easily programmed through standard optimization packages. The proposed algorithm can be adapted for solving the related global optimization problems. This extension of the proposed algorithm is under current investigation by the authors.

References

Ashtiani, A. M. and Ferreira, P. A. V. (2011), On the solution of generalized multiplicative extremum problems, *Journal of Optimization Theory and Applications*, 49, 411–419.

Benson, H. P. (2002), Using concave envelopes to globally solve the nonlinear sum of ratios problem, *Journal of Global Optimization*, 22, 343–364.

Konno, H., Kuno, T. and Yajima, Y. (1994), Global minimization of a generalized convex multiplicative function, *Journal of Global Optimization*, 4, 47–62.

Kuno, T. (2001), A finite branch–and–bound algorithm for linear multiplicative programming, *Computational Optimization and Applications*, 20, 119–135.

Lasdon, L. S., Optimization Theory for Large Systems. MacMillan Publishing Co., New York, 1970.

Li, H-L., Tsai, J-F. and Floudas, C. A. (2008), Convex underestimation for posynomial functions of positive variables, *Optimization Letters*, 2, 333–340.

Oliveira, R. M., Ferreira, P. A. V. (2008), A convex analysis approach for convex multiplicative programming, *Journal of Global Optimization*, 41, 579–592.

Oliveira, R. M., Ferreira, P. A. V. (2010), An outcome space approach for generalized convex multiplicative programs, *Journal of Global Optimization*, 47, 107–118.

Ryoo, H–S. and Sahinidis, N. V. (2003), Global optimization of multiplicative programs, *Journal of Global Optimization*, 26, 387–418.

Schaible, S. and Sodini, C. (1995), Finite algorithm for generalized linear multiplicative programming, *Journal of Optimization Theory and Applications*, 87, 441–455

Thoai, N.V. (1991), global optimization approach for solving the convex multiplicative programming problem, *Journal of Optimization Theory and Applications*, 1, 341–357.

Yu, P. L., Multiple-Criteria Decision Marketing, Plenum Press, New York, 1985.