LP-BASED HEURISTIC FOR PACKING CIRCULAR-LIKE OBJECTS IN A RECTANGULAR CONTAINER

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ABSTRACT

A problem of packing unequal circles in a fixed size rectangular container is considered. An heuristic is proposed, based in a linear relaxation of a integer mathematical model as approximation of the optimal solution of the problem. The circle is considered in a general sense, as a set of points that are all the same distance (not necessary Euclidean) from a given point. The aim is to maximize the (weighted) number of circles placed into the container or minimize the waste. An integer formulation is proposed using a grid approximation of the container and considering the grid nodes as potential positions for centers of the circles. The binary variables represent the assignment of centers to the nodes of the grid. The packing problem is then stated as a large scale linear 0-1 optimization problem. Valid inequalities are used to strengthening the formulation. Numerical results on packing circles and octagons are presented to demonstrate the efficiency of the proposed approach.

Keywords. Circle packing. Integer programming. Large scale optimization.

Main area. OC – Combinatorial Optimization

1. Introduction

Packing problems generally consist of packing a set of items of known dimensions into one or more large objects or containers to minimize a certain objective (e.g. the unused part of the objects or waste). Packing problems constitute a family of natural combinatorial optimization problems applied in computer science, industrial engineering, logistics, manufacturing and production processes (E. Baltacioglu, J.T. Moore and Hill R.R (2013), I. Castillo, F.J. Kampas and J.D. Pinter (2008), H.J. Frazer and J.A. George (1996))

The circle packing problem is a well studied problem (E.G. Birgin and J.M. Gentil (2010), J.A. George (1996), M. Hifi and R. M'Hallah (2009)) whose aim is packing a certain number of circles, each one with a fixed known radius inside a container. The circles must be totally placed in the container without overlapping. The shape of the container may vary from a circle, a square, a rectangular, etc.

Along with aforementioned applications, circle (sphere) packing problems arise in automated radiosurgical treatment planning for treating brain and sinus tumours (J. Wang (1999))]. Radiosurgery uses the gamma knife to deliver a set of extremely high dose ionizing radiation, called "shots" to the target tumor area. For large target regions multiple shots of different intensity are used to cover different parts of the tumor. However, this procedure may result in large doses due to overlap of the different shots. Optimizing the number, positions and individual sizes of the shots can reduce the dose to normal tissue and achieve the require coverage.

Many variants of packing circular objects in the plane have been formulated as nonconvex (continuous) optimization problems with decision variables being coordinates of the centres. The nonconvexity is mainly provided by no overlapping conditions between circles. These conditions typically state that the Euclidean distance separating the centres of the circles is greater than a sum of their radii. The nonconvex problems can be tackled by available nonlinear programming (NLP) solvers, however most NLP solvers fail to identify global optima. Thus, the nonconvex formulation of circular packing problem requires algorithms which mix local searches with heuristic procedures in order to widely explore the search space. It is impossible to give a detailed overview on the existing solution strategies and numerical results within the framework of a single short paper. We will refer the reader to review papers presenting the scope of techniques and applications for the circle packing problem (see, H. Akeb and M. Hifi (2013), E.G. Birgin and J.M. Gentil (2010), C.O. Lopez and J.E. Beasley (2011 and 2013), Y.G. Stoyan and G.N. Yaskov (2013) and the references therein).

In this paper we study packing circular-like objects using a regular grid to approximate the container. The circular-like object is considered in a general sense, as a set of points that are all the same distance (not necessary Euclidean) from a given point. Thus different shapes, such as ellipses, rhombuses, rectangles, octagons can be treated the same way by simply changing the norm used to define the distance. The nodes of the grid are considered as potential positions for assigning centers of the circles. The packing problem is then stated as a large scale linear 0-1 optimization problem. Valid inequalities are used to strengthening the original formulation and improve LP-bounds. Reduced costs of the LP-solution are used to fix some variables in the original problem to get an approximate integer solution. Numerical results on packing circles and regular octagons are presented to demonstrate efficiency of the proposed approach.

To the best of our knowledge, the idea to use a grid was first implemented by Beasley (1985) in the context of cutting problems. Recently this approach was applied for S.I. Galiev and M.S. Lisafina

(2013), I. Litvinchev and L. Ozuna (2013 and 2014), Toledo et al (2013) for packing problems. This work is a continuation of I. Litvinchev and L. Ozuna (2014).

2. The Model

Suppose we have non-identical circles C_k of known radius R_k , $k \in K = \{1,2,...K\}$. Here we consider the circle as a set of points that are all the same distance R_k (not necessary Euclidean) from a given point. In what follows we will use the same notation C_k for the figure bounded by the circle $C_k = \{z \in \mathbb{R}^2 : \|z - z_{0k} \le R_k\|\}$ assuming that it is easy to understand from the context whether we mean the curve or the figure. Denote by S_k the area of C_k. Let at most M_k circles C_k are available for packing and at least m_k of them have to be packed. Denote by $i \in I = \{1,2...,n\}$ the node points of a regular grid covering the rectangular container. Let $F \subseteq I$ be the grid points lying on the boundary of the container. Denote by d_{ij} the distance (in the sense of norm used to define the circle) between points i and j of the grid. Define binary variables $x_i^k = 1$ if centre of a circle C_k is assigned to the point i ; $x_i^k = 0$ otherwise.

In order to the circle C_k assigned to the point i be non-overlapping with other circles being packed, it is necessary that $x_j^l = 0$ for $j \in I,\ l \in K$, such that $d_{ij} < R_k + R_l$. For fixed i,k let $N_{ik} = \{j,l : i \ne j, d_{ij} < R_k + R_l\}$. Let n_{ik} be the cardinality of N_{ik} : $n_{ik} = |N_{ik}|$. Then the problem of maximizing the area covered by the circles can be stated as follows:

$$\max \sum_{i\in I}\sum_{k\in K} S_k x_i^k \tag{1}$$

subject to

$$m_k \le \sum_{i\in I} x_i^k \le M_k, \quad k \in K, \tag{2}$$

$$\sum_{k\in K} x_i^k \le 1, \quad i \in I \setminus F\ , \tag{3}$$

$$R_k x_i^k \le \min_{j\in F} d_{ij}, \quad i \in I, k \in K, \tag{4}$$

$$x_i^k + x_j^l \le 1,\ \text{for}\ i \in I\ , k \in K\ ,\ (j,l) \in N_{ik} \tag{5}$$

$$x_i^k \in \{0,1\}, \quad i \in I, k \in K \tag{6}$$

Constraints (2) ensure that the number of circles packed is between m_k and M_k; constraints (3) that at most one centre is assigned to any grid point; constraints (4) that the point i can not be a centre of the circle C_k if the distance from i to the boundary is less than R_k ; pair-wise constraints (5) guarantee that there is no overlapping between the circles; constraints (6) represent the binary nature of variables.

Similar to plant location problems [21] we can state non-overlapping conditions in a more compact form. Summing up pair-wise constraints (5) over $(j,l) \in N_{ik}$ we get

$$n_{ik} x_i^k + \sum_{j,l \in N_{ik}} x_j^l \le n_{ik}\ \text{ for }\ i \in I\ , k \in K \tag{7}$$

Note that constraints similar to (7) were used in [8] for packing equal circles.

Proposition 1 [13, 15]. Constraints (5), (6) are equivalent to constraints (6), (7).

Thus the problem (1)-(6) is equivalent to the problem (1)-(4), (6), (7). To compare two equivalent formulations, let

$$P_1 = \{x \geq 0 : x_i^k + x_j^l \leq 1, \text{ for } i \in I, k \in K, (j,l) \in N_{ik}\},$$

$$P_2 = \{x \geq 0 : n_{ik} x_i^k + \sum_{j,l \in N_{ik}} x_j^l \leq n_{ik} \text{ for } i \in I, k \in K\}.$$

Proposition 2 [13, 15]. $P_1 \subset P_2$.

As follows from Proposition 2, the pair-wise formulation (1)-(6) is stronger (in the sense of [17]) than the compact one. Numerical experiments presented in [14, 15] demonstrate that the pair-wise formulation is also computationally more attractive since it provides a tighter LP-bound. Bearing in mind these reasons we restrict ourselves by considering below only pair-wise formulations.

By the definition, $N_{ik} = \{j,l : i \neq j, d_{ij} < R_k + R_l\}$ and hence if $(j,l) \in N_{ik}$, then $(i,k) \in N_{jl}$. Thus a half of the constraints in (5) are redundant:

$$x_i^k + x_j^l \leq 1, \text{ for } i \in I, k \in K, (j,l) \in N_{ik}$$

and

$$x_j^l + x_i^k \leq 1, \text{ for } j \in I, l \in K, (i,k) \in N_{jl}.$$

The redundant constraints can be eliminated without changing the quality of LP-bound giving a reduced pair-wise non overlapping formulation. In what follows we will assume that the redundant constraints are eliminated from (5).

Note that all constructions proposed above, including Propositions 1,2, remain valid for any norm used to define a circular-like object. In fact, changing the norm affects only the distance d_{ij} used in the definitions of the sets N_{ik}, Ω_{ik} in the non-overlapping constraints (5). That is, by simple pre-processing we can use the basic model (1)-(6) for packing different geometrical objects of the same shape. It is important to note that the non-overlapping condition has the form $d_{ij} \geq R_k + R_l$ no matter which norm is used.

For example, a circular object in the maximum norm $\|z\|_\infty := \max_i \{|z_i|\}$ is represented by a square, taxicab norm $\|z\|_1 := \sum |z_i|$ yields a rhombus. In a similar way we may manage rectangles, ellipses, etc. Using a superposition of norms we can consider more complex circular objects. For

$$\|z\| := \max_i \{|z_i|, \gamma \sum |z_i|\} \tag{8}$$

and a suitable $0.5 < \gamma < 1$ we get an octagon, an intersection of a square and a rhombus. In particular for $\gamma = 1/\sqrt{2}$ we get a regular octagon.

3. LP-based heuristic

We may expect that the linear programming relaxation of the problem (1)-(6) provides a poor upper bound for the optimal objective. For example, for $K = 1$ and suitable M_k, m_k the point $x_i^k = 0.5$ for all $i \in I$ may be feasible to the relaxed problem with the corresponding objective growing linearly with respect to the number of grid points.

To tightening the LP-relaxation for (1)-(6) we consider valid inequalities aimed to ensure that no grid point is covered by two circles. Define matrix $\left[\alpha_{ij}^k\right]$ as follows. Let $\alpha_{ij}^k = 1$ for $d_{ij} < R_k$, $\alpha_{ij}^k = 0$ otherwise. By this definition, $\alpha_{ij}^k = 1$ if the circle C_k centred at i covers point j. The following constraints ensure that no points of the grid can be covered by two circles:

$$\sum_{k\in K}\sum_{j\in I}\alpha_{ij}^k x_j^k \le 1, \quad i \in I. \tag{9}$$

Note that (9) is not equivalent to non-overlapping constraints (5). Constraints (9) ensure that there is no overlapping in grid points, while (5) guarantee that there is no overlapping at all. We can treat (9) as a relaxed non-overlapping conditions and expect that refining the grid reduces overlapping. The valid inequalities (9) hold for any norm used to define the circular object.

Numerical experiments presented in [14, 15] demonstrate that aggregating valid inequalities (9) to the original problem (1)-(6) improves significantly the value of the corresponding LP-bound. Moreover, valid inequalities change the structure of the optimal LP-solution. Below, we will use the same term LP relaxation (LP-bound) for the problem (1)-(5) as well as for problems (1)-(5), (9) and (1)-(4), (9). That is along with relaxing integrality constraints (6) we may substitute non-overlapping conditions (5) for valid inequalities (9).

Let G be a set of the nodes of an original grid and FG be a set of the nodes of the refined grid constructed such, that $G \subseteq FG$, i.e. all nodes of the original grid remain the nodes of the refined one. Let z_G and z_{FG} be the optimal values of the integer problems (1)-(6) or (1)-(6), (9) obtained for corresponding grids G, FG. Then we have

$$z_G \le z_{FG} \le lp_{FG}, \tag{10}$$

where lp_{FG} is the value of the LP-bound corresponding to the grid FG. Here the first inequality holds since we may construct a feasible solution to the problem corresponding to FG by setting $x_i = 0$ for $i \in FG - G$ and leaving all other components equal to G - optimal solution. Thus we can construct LP-bounds for the original objective using grids different from the original one.

Suppose that a relaxed problem for the grid FG is solved and corresponding reduced costs are known. The heuristic algorithm below aimed to reduce the number of variables in integer problem (1)-(6), (9) by fixing $x_i = 0$ for the nodes of G with sufficiently negative reduced costs.

LP-based heuristic.

Step 1. For the original grid G define a refined grid FG, $G \subseteq FG$, and solve LP-relaxation for the grid FG. Let d_i, $i \in FG$ be corresponding reduced costs.

Step 2. Define the set of non-positive reduced costs, $FG_- = \{i \in FG : d_i \le 0\}$.

Step 3. For $i \in FG_-$ define scaled reduced costs $\bar{d}_i \in [0,1]$ as follows: $\bar{d}_i = |d_i| / (\max_{i\in FG_-} |d_i|)$

Step 4. For a fixed parameter $\delta \in (0,1)$ define a set of "sufficiently" negative reduced costs: $FG_{\delta-} = \{i \in FG_- : \bar{d}_i \ge \delta\}$

Step 5. Solve the integer problem (1)-(6), (9) corresponding to the grid G fixing $x_i = 0$ for $i \in FG_{\delta-} \cap G$.

4. **Computational results**

In this section we numerically compare LP-relaxations obtained for different grids and study the impact of introducing valid inequalities for the case of packing equal circular objects without the limits (2) for the availability of the objects. A rectangular uniform grid of size Δ (Δ is the space between nodes of the grid and is equal for both "x" and "y" axes) along both sides of the container was used to form an initial grid. The test bed set of 9 instances from [8, Table 3] was used for packing maximal number of objects into a rectangular container of width 3 and height 6. All optimization problems were solved by the system CPLEX 12.6. The runs were executed on a desktop computer with CPU AMD FX 8350 8-core processor 4 Ghz and 32Gb RAM.

First, we compare linear programming bounds obtained by different grids for circular objects defined by the Euclidian norm (circles). The LP-bound was calculated for the problem (1)-(4), (9), that is non-overlapping constraints (5) were substituted for valid inequalities (9). The following three grids were used: original grid of size Δ generated the same way as in [8, Table 3] with n_Δ nodes and two refined grids with $n_{\Delta/2}$ and $n_{\Delta/3}$ nodes obtained by reducing the original grid size to $\Delta/2$ and $\Delta/3$, respectively. The results of the numerical experiment are given in Table 1. Here the first three columns show the characteristics of the instances, number of instance, radius R used to define the circular object and original size of the grid Δ , the four column show the optimal integer solution z_I obtained for the grid Δ. The rest of the columns give the number of grid points (n), value of the corresponding LP-relaxation (z_{LP}) and CPU time (in seconds).

#	R	Δ	z_I	n_Δ	z_{LP}	CPU	$n_{\Delta/2}$	z_{LP}	CPU	$n_{\Delta/3}$	z_{LP}	CPU
1	0.5	0.125	**18**	697	**19**	0	2673	**18.06**	3	8017	**18.14**	23
2	0.625	0.078125	**10**	1403	**10**	1	5445	**10**	38	16333	**10**	390
3	0.5625	0.0625	**13**	2449	**14.07**	5	9577	**13.96**	130	28729	**13.7**	1500
4	0.375	0.09375	**32**	1425	**36.33**	0	5537	**34.54**	10	16609	**34.75**	88
5	0.3125	0.078125	**45**	2139	**53.4**	1	8357	**50.76**	23	25069	**50.77**	350
6	0.4375	0.546875	**21**	3666	**23.86**	5	14399	**24.01**	200	43195	**24.19**	3400
7	0.25	0.0625	**74**	3649	**90.98**	2	14337	**85.76**	180	43009	**85.24**	3100
8	0.275	0.06875	**61**	2880	**72**	2	11289	**67.78**	70	33865	**67.52**	1100
9	0.1875	0.046875	**140**	6897	**152.9**	35	27233	**151.8**	3401	81697	**151.8**	3591

Table 1. LP-bounds for circles

In the second part of the experimentation, we compare the best bounds obtained using the original model integer with the bounds obtained with the heuristic, the objective is prove if the bounds obtained with the heuristic are of quality similar. Table 2 provides results obtained by the heuristic proposed in the previous section for $\delta = 0.1$ and the grid with $n_{\Delta/2}$ nodes used to form the relaxed problem. Here the first four columns present instance number, number of integer variables n_Δ corresponding to the original grid, optimal integer solution z_I and the corresponding CPU time. For all integer problems in Table 2 *mipgap* = 0 was set for running CPLEX. Computations to get integer solution z_I were interrupted after the computational time exceeded 12 hours CPU time and the value in parenthesis gives the corresponding *mipgap*. More details on getting z_I one can find in [15]. The last three columns give the number of integer variables in the reduced problem ($n_{reduced}$), corresponding integer solution (z_H) and CPU time. For the heuristic the time limit was set to 1 hour CPU.

#	n_Δ	z_I	CPU	$n_{reduced}$	z_H	CPU
1	697	**18**	1	404	**18**	0
2	1403	**10**	41	718	**10**	4
3	2449	**13**	186	738	**13**	9
4	1425	**32**	4	790	**32**	1
5	2139	**45**	114	1372	**45**	28
6	3666	**21**	17654	1441	**21**	130
7	3649	**74** (5%)	> 12 h.	2491	**74**	1400
8	2880	**61**	177	777	**61**	24
9	6897	**140** (5%)	> 12 h.	558	**139** (1.3%)	3600

Table 2. Heuristic solutions for circles

Tables 3, 4 present LP-bounds and heuristic solutions obtained for packing regular octagons corresponding to $\gamma = 1/\sqrt{2}$ in (7). In both Tables CPU time was limited to 1 hour.

#	R	Δ	z_I	n_Δ	z_{LP}	CPU	$n_{\Delta/2}$	z_{LP}	CPU	$n_{\Delta/3}$	z_{LP}	CPU
1	0.5	0.125	**18**	697	**19**	0	2673	**18**	12	8017	**18**	20
2	0.625	0.078125	**9**	1403	**10**	1	5445	**9.524**	37	16333	**9.571**	450
3	0.5625	0.0625	**12**	2449	**14.0743**	3	9577	**13.18**	140	28729	**13.00**	1700
4	0.375	0.09375	**26**	1425	**30.9485**	0	5537	**30.94**	9	16609	**30.94**	80
5	0.3125	0.078125	**41**	2139	**53.4043**	1	8357	**50**	19	25069	**49.79**	250
6	0.4375	0.546875	**20**	3666	**22.5537**	5	14399	**22.75**	200	43195	**22.84**	2800
7	0.25	0.0625	**72**	3649	**90.9767**	2	14337	**81.57**	71	43009	**83.99**	170
8	0.275	0.06875	**50**	2880	**59.014**	2	11289	**59.03**	67	33865	**59.21**	570
9	0.1875	0.046875	**106**	6897	**134.342**	25	27233	**134.0**	2800	81697	**134.0**	3487

Table 3. LP-bounds for octagons

As we can see from Tables 1, 3 refining the grid typically (but not always) results in improving the LP-bound. However, solving LP-relaxation for fine grids may be computationally too expensive. Concerning the quality of the integer solution obtained by the heuristic, we may conclude that in most cases (except for the instance 9 for packing circles) the optimal solution was obtained. The use of heuristic reduces CPU time significantly.

#	n_Δ	z_I	CPU	$n_{reduced}$	z_H	CPU
1	697	**18**	1	371	**18**	0
2	1403	**9**	52	692	**9**	2
3	2449	**12**	202	752	**12**	5
4	1425	**26**	49	830	**26**	1
5	2139	**41**	6850	1383	**41**	77
6	3666	**20**	1430	1795	**20**	59
7	3649	**72**	22	2261	**72**	25
8	2880	**50**	20495	2254	**50** (4.8%)	3600
9	6897	**106**	> 12 h.	5652	**106** (5.7%)	3600

Table 4. Heuristic solutions for octagons

Figures 1-4 present optimal packing and grid points left after heuristic node-reduction based on reduced costs.

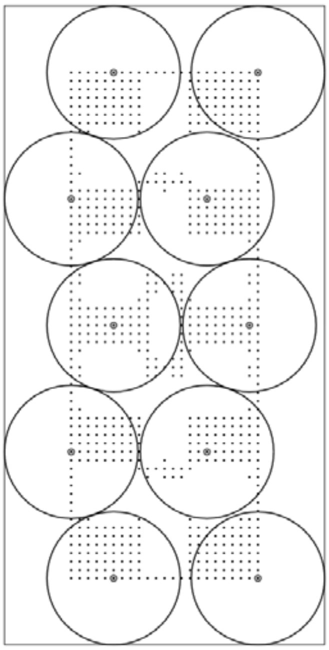

Fig.1. Circles, instance 2

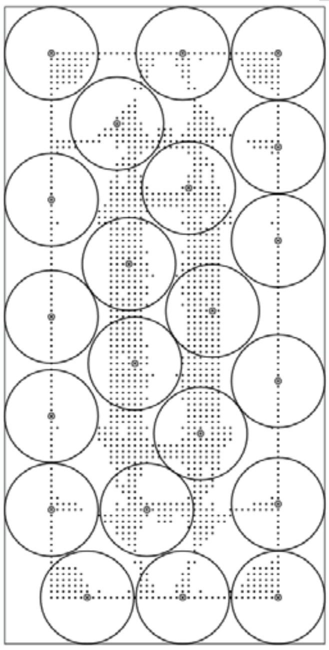

Fig.2. Circles, instance 6

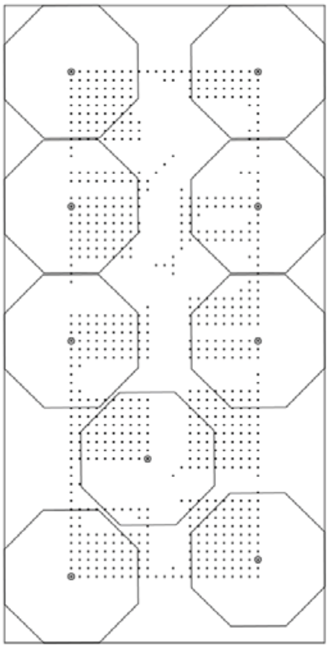

Fig.3. Octagons, instance 2

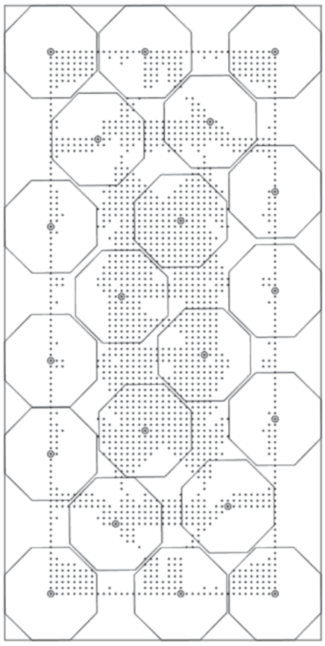

Fig.4. Octagons, instance 6

4. Conclusions

An integer formulation and LP-based heuristic were proposed for approximated packing circular-like objects in a rectangular container. Different shapes of the objects, such as circles, ellipses, rhombuses, rectangles, octagons can be considered by simply changing the norm used to define the distance. The presented approach can be easily generalized to the three (and more) dimensional case and to different shapes of the container, including irregulars. Valid inequalities are proposed to strengthening the original formulation. A heuristic approach is proposed based on analysis of the reduced costs obtained by LP-relaxation. An interesting direction for the future research is to study the use of Lagrangian relaxation and corresponding heuristics (see I. Litvinchev, S. Rangel and J. Saucedo (2010)) to cope with large dimension of arising problems.

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