A systematic review of heuristics for symmetric-matrix bandwidth reduction: methods not based on metaheuristics

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ABSTRACT

Heuristics for bandwidth reduction of matrices is used to reduce computational and storage costs of resolution of large sparse linear systems. Bandwidth reduction consists of carrying out permutations of lines and columns so that the coefficients remain near the main diagonal. When considering an adjacency matrix of a graph, bandwidth reduction can be considered in the sense of modifying the order in which the graph vertices are numbered. In this study, heuristics for bandwidth reduction are revised aiming at determining which of them offers the best bandwidth reduction at a very low computational cost. Specifically, heuristics not based on metaheuristics are reviewed. Moreover, 44 heuristics tested for bandwidth reduction were found. Among them, 4 are recommended as possible state-of-the-art heuristics for addressing the problem.

KEYWORDS. Bandwidth reduction, heuristics, sparse matrices.

Main Area: TAG − Teoria e Algoritmos em Grafos

1. Introduction

Resolution of large sparse linear systems $Ax = b$, in which A is a sparse matrix, is central in various simulations in science and engineering and is generally the part of the simulation that requires the highest computational cost. The main origin of problems with large-scale matrices arises from discretization of elliptical or parabolic partial differential equations (BENZI, 2002). The methods of finite elements, finite differences and finite volumes are some of the most popular numerical methods for problem solving related to physical phenomena that are modeled by partial differential equations. Large sparse linear systems are produced when these methods are applied. On the other hand, large sparse linear systems are also originated from problems that are not modeled by partial differential equations such as chemical engineering processes, design and analysis of integrated circuits, and power system networks (BENZI, 2002). Benzi (2002) described the main fields in which large sparse linear systems are required to be solved.

One needs a great deal of memory space and high processing cost to store and to solve these large-scale linear systems. For example, Kaveh (2004, p. 221) described that in structural mechanics, ranging from 30% to 50% of the computational cost can be associated to solving linear systems for real-world problems and this computational cost may achieve 80% in non-linear problems for optimization of structures.

The current hierarchical memory architecture and the paging policies favor programs that consider locality of reference. Thus, a major concern when projecting a new algorithm should be considering cache coherence, i.e., in the address memory space, a sequence of recent memory references is clustered locally rather than randomly. A way of projecting an algorithm to obtain a sequence of graph vertices with cache coherence is with heuristics for bandwidth reduction. Therefore, heuristics for bandwidth reduction are used to obtain low computational and storage costs for solving large sparse linear systems.

Let $A = [a_{ij}]$ be a symmetric matrix $n \times n$. The bandwidth of line $i \in [1, n]$ is $\beta_i(A) = i - \min(j : (1 \leq j < i) \;\; a_{ij} \neq 0)$. Bandwidth $\beta(A)$ is the largest distance between the non-null coefficient of the lower triangular matrix and the main diagonal, considering all lines of the matrix, that is, $\beta(A) = \max((1 \leq i \leq n) \; \beta_i(A)) = \max((1 \leq i \leq n) \; (1 \leq j < i) \; (i - j) \mid a_{ij} \neq 0)$.

Heuristics for bandwidth reduction are vertex reordering methods and they can reduce the computational cost of direct and iterative methods for solving linear systems. More specifically, heuristics for bandwidth reduction can benefit the Gaussian elimination (and methods based on it) if the matrix is symmetric. According to Tarjan (1975), the Gaussian elimination requires $O(n \cdot \beta^2)$ operations to solve a linear system with n lines and bandwidth β. Clearly, if $\beta \cong n$, then, $O(n^3)$ operations are carried out. Also, the Gaussian elimination requires $O(n \cdot \beta)$ spaces in memory when a storage system based on vectors is employed. Although direct methods for solving linear systems are superseded by super-nodal and multi-frontal solvers, bandwidth reduction can benefit implemented applications with direct methods. On the other hand, iterative methods are used when scalability is a crucial aspect of the application. A prominent method for solving large sparse linear systems is the Conjugate Gradient Method (HESTENES; STIEFEL, 1952; LANCZOS, 1952). One can obtain considerable reduction in the computational cost of this method by applying a local ordering of the vertices (DUFF; MEURANT, 1989) of the corresponding graph of A in order to improve cache hit rates. This local ordering can be obtained by applying a heuristic for bandwidth reduction (BURGESS; GILES, 1997; DAS et al., 1992). Moreover, Benzi, Szyld and Duin (1999) showed that heuristics for bandwidth reduction can benefit the computational cost of the Generalized minimal residual (GMRES) method (SAAD; SCHULTZ, 1986).

The bandwidth minimization problem seems that began in the 1950s when structural engineers investigated the computational resolution of large sparse linear systems and were already concerned about the bandwidth of the matrices - for example, see Livesley (1960) and references cited therein. The bandwidth minimization problem can be understood as finding the smallest bandwidth of a symmetric matrix. Papadimitriou (1976) showed that this problem is hard. When an arbitrary integer k and a graph G are given, Garey et al. (1978) demonstrated that the following decision problem is NP-complete: whether or not there exists an ordering of the vertices such that the adjacency matrix of G has bandwidth $\leq k$, even when G is defined by the class of free trees with all vertices of degree ≤ 3. Checking all $n!$ possible sequences associated to a matrix of a sufficient large n is unfeasible. Thus, many heuristics have been proposed for the bandwidth, profile and/or wavefront reduction since the 1960s. These problems are interrelated but independent. Bandwidth reduction of a matrix is carried out by means of permutations of its lines and columns in a way that allows the non-null coefficients to remain near the main diagonal. Certainly, when considering the adjacency matrix of a graph, bandwidth reduction can be regarded as altering the order in which the vertices of this graph are numbered. For example, vertices of graph shown in Figure 1 are renumbered in Figure 2 so that the bandwidth of its adjacency matrix is reduced.

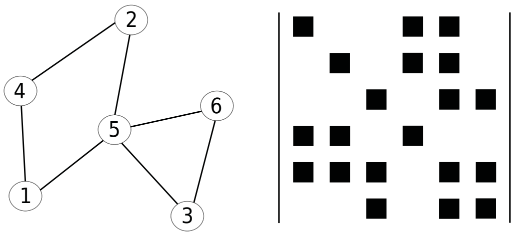

Figure 1: A graph and its adjacency matrix.

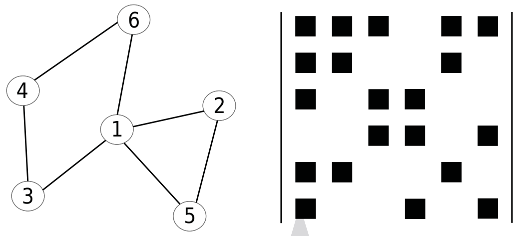

Figure 2: Vertices of graph shown in Figure 1 renumbered and its adjacency matrix.

As many heuristics for bandwidth reduction have been published, finding out which of them is the best one is not a simple task because of the characteristics of the problem to be solved such as: size and sparseness of the matrix, disposal and distribution of the non-null coefficients, required memory, profile of the matrix, envelope, and bandwidth (BERNARDES; GONZAGA DE OLIVEIRA, 2015). The large number of heuristics available means that the user has a difficult task to decide which heuristic to apply. Several comparisons among heuristics have been published; but only for a few of them. In addition, there have been few reviews published on this matter. Cuthill (1972) carried out a comparative study among results of the heuristics known up to 1971. In 1976, Gibbs, Poole and Stockmeyer (1976) made comparisons between the results of 6 heuristics. Benzi (2002) reviewed preconditioning techniques for iterative solution of large sparse linear systems. This review focused on techniques to improve performance and reliability of Krylov subspace methods. However, this review was not strictly on heuristics for bandwidth reduction. More recently, a systematic review of metaheuristic-based heuristics for bandwidth reduction was addressed by Chagas and Gonzaga de Oliveira (2015). In addition, a systematic review of heuristics for profile (and envelope) reduction was addressed by Bernardes and Gonzaga de Oliveira (2015).

Some heuristics for this problem reduce the bandwidth significantly; however, these heuristics may demand considerable computational cost to accomplish the task. It should be noted that the benefits obtained in the total computational cost in solving a large sparse linear system after the matrix bandwidth reduction should not be impaired by the high computational costs required to perform the proper bandwidth reduction. In addition, a large bandwidth reduction may result in a small computational cost reduction incurred for solving the a linear system. Thus, this systematic investigation of heuristics for bandwidth reduction is limited to methods that are not based on metaheuristic. It was undertaken aiming at determining heuristics that present reasonable bandwidth reduction at very low computational cost because a reasonable bandwidth reduction by a very fast heuristic may be better than a very large bandwidth reduction provided by a heuristic with high computational cost.

This present study is structured as follows. Section 2 is concerned about how the systematic inquiry was carried out in this study. Section 3 addresses simulations performed by researchers among results of heuristics not based on metaheuristic. Section 4 outlines the best possible heuristics for bandwidth reduction regarding to the best benefits in terms of bandwidth reduction at a very low computational cost. Section 5 concludes this study with final considerations and recommendations for future studies.

2. Systematic review

Similarly to the review of metaheuristic-based heuristics for bandwidth reduction presented by Chagas and Gonzaga de Oliveira (2015), this present review began in May, 2014, and concerns heuristics for bandwidth reduction that are not based on metaheuristics. It was conducted with the aid of Scopus® and Google Scholar databases.

A search was performed in Scopus® database using the terms: TITLE-ABS-KEY((matrix bandwidth reduction) or (matrix bandwidth minimization)). These terms were searched in the title, abstract and keywords of the articles indexed in the database. This search resulted in 502 articles. Among these articles retrieved, heuristics for bandwidth reduction designed by techniques that are not based on metaheuristics were selected. The titles and abstracts of the articles found were then read independently by two reviewers and as there were no disagreements in the selections made, a third reviewer was not needed. As well as the articles that met the eligibility criteria, other articles were analyzed to support some of the concepts involved in the heuristics that had been identified (CHAGAS; GONZAGA DE OLIVEIRA, 2015).

In addition, Google Scholar database was also searched to find additional publications related to specific heuristics. Then, to have a clear comparison of the studies selected, data were extracted according to the following headings: authors, year of publication, results and conclusions (CHAGAS; GONZAGA DE OLIVEIRA, 2015). Among the 502 articles retrieved, 16 heuristics that are not based on metaheuristics were selected and are shown in Table 1.

Article	Heuristic
Grooms (1972)	Grooms
Collins (1976)	Collins
Konishi, Shiraishi and Taniguchi (1976)	Konishi-Shiraishi-Taniguchi
Puttonen (1983)	Puttonen
Recuero and Gutierrez (1984)	R&G1
	R&G2
Burgess and Lai (1986)	Burgess-Lai
Kaveh (1986)	Four-steps
Luo (1992)	Luo
Scott and Han (1994)	SHR
	SHG
Esposito et al. (1998)	WBRA
Del Corso and Romani (2001)	Del Corso-Romani
Wang, Guo and Shi (2009)	GGPS
Glüge (2010)	Glüge
Doss and Arathi (2011)	Doss-Arathi

Table 1: 16 heuristics not based on metaheuristics found by using Scopus® database.

After this, a backward citation tracking was performed based on the articles found. As well as the 502 retrieved articles, other 21 heuristics that are not based on metaheuristic were found. This means that some of the 502 articles cited other articles in which other heuristics were proposed. These 21 heuristics are listed in Table 2. These cited articles were naturally analyzed too. In addition, 3 other heuristics not based on metaheuristics were found in the second-phase backward citation tracking. These heuristics are listed in Table 3.

It should be noted that the publication of GowriSankaran, Miller and Opatrny (1990) (cited in Dueck and Jeffs (1995)) in particular is not a heuristic for bandwidth reduction. Despite its title, GowriSankaran, Miller and Opatrny (1990) proposed two algorithms for generating level structures from trees.

Article retrieved from Scopus®	Heuristic cited
Arathi, Doss and Kanakadurga (2012)	Alway and Martin (1965)
	Rosen (1968)
	CM (CUTHILL; MCKEE, 1969)
	RCM (GEORGE, 1971)
	GPS (GIBBS; STOCK; STOCKMEYER, 1976)
	GowriSankaran and Opatrny (1990)
Gibbs, Poole and Stockmeyer (1976)	King (1970)
	Levy (1971)
	Wang (1973)
Lim, Rodrigues and Xiao (2006)	Arany, Martin and Szoda (1971)
Scott and Han (1994)	Cheng (1973)
	NSAS (ARMSTRONG, 1984)
Recuero and Gutierrez (1984)	Rodrigues (1975)
Smyth (1985)	Smyth and Arany (1976)
Wang, Guo and Shi (2009)	Gibbs-King (GIBBS, 1976)
	Boutora et al. (2007)
Kaveh and Roosta (1999)	Razzaque (1980)
Doss and Arathi (2011)	Taniguchi and Shiraishi (1980)
Mamaghani and Meybodi (2011)	RCM-GL (GEORGE; LIU, 1981)
Burgess and Lai (1986)	Martin and Alarcon (1982)
Koohestani and Poli (2011)	*Spectral* (BARNARD; PHOTEN; SIMON, 1995)

Table 2: 21 heuristics not based on metaheuristics found in the first-phase backward citation tracking in the 502 articles retrieved from Scopus® database.

Article retrieved from Scopus®	Cited article	Heuristic found
Lai and Chang (2004)	Snay (1976)	Akyuz and Utku (1968)
Gibbs, Poole and Stockmeyer (1976)	Levy (1971)	Melosh and Bandford (1969)
Kaveh and Roosta (1999)	Everstine (1979)	Akhras and Dhatt (1976)

Table 3: 3 other heuristics not based on metaheuristics found in a second-phase backward citation tracking.

Simulations and comparisons related to these 40 heuristics were analyzed. As a result of this analysis, no other article was found that showed simulations and comparisons in a manner that could suggest that 4 heuristics (RCM-GL, Burgess-Lai, Wonder Bandwidth Reduction Algorithm (WBRA), and Generalized GPS (GGPS)) might be considered to be surpassed by any other heuristic. Apart from these 4 heuristics, no other simulation or comparison showed that the heuristic of Razzaque (1980) could be superseded by any other heuristics in the articles analyzed. Thus, these 5 heuristics were searched in Google Scholar by: "paper's title" "authors' surname" "year" matrix bandwidth reduction. The reason for this was to find other articles that had simulations and comparisons related to these 5 heuristics. In this search, the heuristic of Santos and Groehs (1988) was found. This heuristic arose from the publication of Razzaque (1980). Thus, 41 heuristics were found at this stage of this systematic review. When the heuristic of Santos and Groehs (1988) was found, our interest was to find out if there were any other heuristics for bandwidth reduction published in that journal. Thus, we searched in the journal Mecánica Computacional (http://www.cimec.org.ar/ojs/index.php/mc/index) by: "Minimização de largura de banda". As a result of this, the heuristics of Ramalho and Corrêa (1991) and of Serra (1997) were found. In turn, when studying this article, the heuristic of Teixeira and Groehs (1991) was found. Thus, a total of 44 heuristics for bandwidth reduction were found in this systematic review. It should be noted that these 44 heuristics are not based on metaheuristics.

3. Comparisons of results of heuristics that are not based on metaheuristics

In this section, simulations performed by researchers among results of heuristics that are not based on metaheuristic are addressed. Similarly to the analysis carried out in Chagas and Gonzaga de Oliveira (2015), the GPS heuristic was employed as a benchmark, but it was not the only one used in the assessments and comparisons carried out among the results of the heuristics for bandwidth reduction. The GPS heuristic computes bandwidth reduction at a low computational cost. Since it is a classic heuristic for bandwidth reduction, it has been widely employed by researchers to compare the results of heuristics in this matter. Details of the assessments and comparisons are described below.

The 44 heuristics for bandwidth reduction found that are not based on metaheuristics are listed in the first column of Table 4. In their work, the heuristics' authors compared results of their heuristics with results of other heuristics. Those comparisons made by the authors were studied seeking to identify heuristics that might have been surpassed by other heuristics. In the second column of Table 4, it is shown heuristics whose results (bandwidth reduction) surpassed (without considering computational cost) or showed comparable results but with lower computational cost than the heuristic shown in the first column. These tests have been outlined in publications shown in the third column of Table 4. Heuristics listed in Table 4 that cannot be directly considered surpassed are commented below.

No other comparisons of results made with the heuristic of Arany, Martin and Szoda (1971) and with the heuristic of GowriSankaran and Opatrny (1990) were found. The papers in which these heuristics were proposed were not found as well. Thus, only the titles and abstracts of these papers were read. Therefore, these heuristics were not included as the possible best for bandwidth reduction.

We found 9 heuristics that outperformed results of the GPS heuristic. The identified heuristics not based on metaheuristics that outperformed results of the GPS heuristic are: NSAS (ARMSTRONG, 1984), heuristics of Burgess and Lai (1986), Luo (1992), SHR (SCOTT; HAN, 1994); SHG (SCOTT; HAN, 1994); RCM with George-Liu algorithm (GEORGE; LIU, 1979) to find a pseudo-peripheral initial vertex (RCM-GL (GEORGE; LIU, 1981)), with tests presented by Esposito et al. (1998); WBRA (ESPOSITO et al., 1998); heuristic of Boutora et al. (2007); and GGPS (WANG; GUO; SHI, 2009).

Smyth and Arany (1976) presented few tests with small instances with their heuristic. They compared results of the proposed heuristic, which in this study is named as SA76, with results presented by the GPS heuristic. One can state that results of the SA76 heuristic were similar to results of the GPS heuristic. As results of the GPS heuristic were considered surpassed by other heuristics, the SA76 heuristic was considered also surpassed.

Recuero and Gutierrez (1984) proposed two heuristics for bandwidth reduction. These two heuristics are named in this present study as R&G1 and R&G2. The results of these heuristics were compared with results of the following heuristics: Cuthill and McKee (1969), of Akyuz and Utku (1968), of Rosen (1968), of Grooms (1972) and with the heuristic of Rodrigues (1975), in 6 small instances ranging from 15 to 52 vertices. The R&G1, R&G2 and the heuristic of Rodrigues (1975) obtained similar results to the CM heuristic, which is considered surpassed by other heuristics. In addition, the R&G1 heuristic may have high computational cost in certain instances, according to the authors. Therefore, these 3 heuristics were not considered as the possible best ones for bandwidth reduction.

The RCM-GL heuristic and the heuristic of Burgess and Lai (1986) are commented in subsection 4 as a possible best heuristics for bandwidth reduction because we did not find any result that provided evidence that these heuristic were surpassed by any other heuristic, taking into account both bandwidth reduction and computational cost. The WBRA heuristic is also commented in subsection 4 as a possible best heuristic for bandwidth reduction because it showed much lower computational cost than the heuristics that surpassed it in bandwidth reduction (see Table 4). The

Heuristic	Year	Surpassed in bandwidth reduction by	Tests in
Alway-Martin	1965	GPS	Gibbs (1980)
Akyuz-Utku	1968	CM, Rodrigues, Recuero-Gutierrez	Recuero and Gutierrez (1984)
		CM, RCM, Collins, Akhras-Dhatt, Teixeira-Groehs and Serra	Serra (1997)
Rosen	1968	CM, Grooms, Rodrigues and Recuero-Gutierrez	Recuero and Gutierrez (1984)
		Melosh-Bandford, CM, RCM, Grooms, Collins, Akhras-Dhatt, Teixeira-Groehs, and Serra	Serra (1997)
CM	1969	RCM	George (1971)
		NSAS	Armstrong (1984)
		Burgess-Lai	Burgess and Lai (1986)
Melosh-Bandford	1969	Collins and Serra	Serra (1997)
King	1970	RCM, Collins and GPS	Gibbs, Poole and Stockmeyer (1976)
		Gibbs-King	Gibbs (1976)
		GPS	Gibbs (1980)
Arany-Smyth-Szoda	1971	-	-
Levy	1971	CM and RCM	Cuthill (1972)
RCM	1971	GPS	Gibbs, Poole and Stockmeyer (1976, 1976)
Grooms	1972	Puttonen	Puttonen (1983)
		CM, Rodrigues, Recuero-Gutierrez,	Recuero and Gutierrez (1984)
		Puttonen, Burgess-Lai,	Burgess and Lai (1986)
		CM, RCM, Collins and Serra	Serra (1997)
Cheng	1973	RCM, Collins and GPS	Gibbs, Poole and Stockmeyer (1976)
Collins	1973	GPS	Gibbs, Poole and Stockmeyer (1976)
		CM and Ramalho-Corrêa	Ramalho and Corrêa (1991)
Wang	1973	RCM, Collins and GPS	Gibbs, Poole and Stockmeyer (1976)
Rodrigues	1975	-	-
Konishi-Shiraishi-Taniguchi[1]	1976	Burgess-Lai	Burgess and Lai (1986)
Smyth-Arany	1976	-	-
Akhras-Dhatt	1976	Collins	Serra (1997)
		Puttonen	Puttonen (1983)
		Puttonen and Burgess-Lai	Burgess and Lai (1986)
GPS	1976	-	-
Gibbs-King	1976	GPS	Gibbs (1976) and Gibbs (1980)
Taniguchi-Shiraishi	1980	CM	Taniguchi and Shiraishi (1980)
Razzaque	1980	CM, Grooms and Santos-Groehs	Santos and Groehs (1988)
RCM-GL	1981	WBRA	Esposito et al. (1998)
		GPHH-band	Koohestani and Poli (2011)
Martin-Alarcon	1982	Burgess and Lai	Burgess and Lai (1986)
Puttonen	1983	Burgess and Lai	Burgess and Lai (1986)
R&G1	1984	-	-
R&G2	1984	-	-
NSAS	1984	SHR	Scott and Han (1994)
Four-steps	1986	CSS-band[2]	Kaveh and Sharafi (2012)
Burgess-Lai	1986	-	-
Santos-Groehs	1988	CM and Grooms	Santos and Groehs (1988)
GowriSankaran-Opatrny	1990	-	-
Ramalho-Corrêa	1991	CM	Ramalho and Corrêa (1991)
Teixeira-Groehs	1991	RCM and Collins	Serra (1997)
Luo	1992	GPS	Luo (1992)
SHG	1994	SHR	Scott and Han (1994)
SHR	1994	TS-band[2,3](MARTÍ et al., 2001)	Scott and Han (1994) and Lim, Lin and Xiao (2007)
Spectral[4]	1995	RCM	Del Corso and Romani (2001)
		GPHH-band[2]	Koohestani and Poli (2011)
Serra	1997	Collins	Serra (1997)
WBRA	1998	ACO-band[2] (LIM et al., 2006)	Lim et al. (2006)
		NS-HC[2] (LIM; RODRIGUES; XIAO, 2006)	Lim, Rodrigues and Xiao (2006)
		PSO-HC[2] (LIM; LIN; XIAO, 2007)	Lim, Lin and Xiao (2007)
Del Corso-Romani	2001	RCM	Del Corso and Romani (2001)
Boutora et al.	2007	TS-band[2,3] (MARTÍ et al., 2001)	Boutora et al. (2007) and Martí et al. (2001)
GGPS	2009	-	-
Glüge	2010	-	-
Doss-Arathi	2011	-	-

[1] The article was not found and probably the heuristic is not based on metaheuristics.
[2] Heuristic not based on metaheuristics; therefore, it was not considered in this work.
[3] Indirect comparisons related to the GPS heuristic.
[4] Heuristic originally designed for profile reduction, but its results were compared to other heuristics for bandwidth reduction.

Table 4: Comparisons among results of heuristics for bandwidth reduction that are not based on metaheuristics. Heuristics without being filled in columns “Surpassed in bandwidth reduction by” and “Tests in” are described in the text.

GGPS heuristic is also commented in subsection 4 because it presented results in bandwidth reduction that stood out compared to other heuristics.

The heuristic of Boutora et al. (2007) is a modified breadth-first search applied to triangular meshes. The executables tested by Boutora et al. (2007) were produced by different compilers: *Fortran 90* for the proposed heuristic and *Fortran* for the GPS heuristic. In addition, Boutora et al. (2007) stated that they chose a vertex at the border of the domain as the initial vertex for their heuristic for bandwidth reduction. A vertex at the border of the domain can be easily obtained in certain finite volume or element discretizations. Additionally, this is beneficial with regard to applying an algorithm to find an initial vertex to begin the ordering of vertices. This may have produced the fastest results of the heuristic of Boutora et al. (2007) compared with results of the GPS heuristic. On the other hand, the heuristic of Boutora et al. (2007) was not considered as a possible best heuristic for bandwidth reduction because it is specific to triangular meshes.

The heuristic of Glüge (2010) was not considered as a possible best heuristic for bandwidth reduction. That is because the author did not compare results of his heuristic with results of any other heuristic. There are heuristics available to be compared, for example, in MATLAB (2015), such as the RCM-GL and Spectral heuristics. The Cuthill and McKee (1969) and RCM-GL heuristics are also available on Boost C++ Libraries (http://www.boost.org/doc/libs/1_58_0/libs/-graph/doc/cuthill_mckee_ordering.html). In addition, in Netlib (*http://www.netlib.org*), there are implementations of the GPS heuristic in *http://www.netlib.org/toms/508* (CRANE et al., 1976), and of the Gibbs-King heuristic in *http://www.netlib.org/509* (GIBBS, 1976). Also in *http://www.netlib.-org/toms/582* (LEWIS, 1982), there are implementations of the GPS and Gibbs-King heuristics.

The heuristic of Doss and Arathi (2011) is specific to L or Z graphs. In tests conducted by Doss and Arathi (2011), this heuristic obtained similar results to the RCM and GPS heuristics. Doss and Arathi (2011) did not show results regarding to computational cost. Since the RCM and GPS are heuristics considered surpassed by other heuristics, the heuristic of Doss and Arathi (2011) was also considered surpassed.

4. Possible best heuristics regarded to bandwidth reduction at a very low computational cost

In this section, heuristics not based on metaheuristic that have been identified as being possibly better for bandwidth reduction are addressed. In order to be considered for inclusion in this section as one of the possible best choices with regard to bandwidth reduction per computational cost, a heuristic must show a reasonable bandwidth reduction at a very low computational cost. This means that a heuristic must yield reasonably better results for bandwidth reduction than the GPS heuristic and must have a computational cost that is lower than, or similar to, the computational cost of the GPS heuristic in direct or indirect comparisons made by researchers in their publications.

In this systematic review, 4 heuristics were identified as the possible state-of-art with regard to bandwidth reduction at a very low computational cost. Comparisons of results of these 4 heuristics are commented below.

• the RCM and GPS heuristics are probably the most well-known heuristics for bandwidth reduction. In addition, based on tests carried out by Gibbs, Poole and Stockmeyer (1976), one can assume that the GPS heuristic surpassed the RCM heuristic. As a result, the GPS heuristic became popular for bandwidth reduction. However, tests conducted by Gibbs, Poole and Stockmeyer (1976) were carried out with the original RCM heuristic (GEORGE, 1971). A variant of the RCM heuristic starting with a pseudo-peripheral vertex given by the George and Liu (1979) algorithm is here named as RCM-GL heuristic (GEORGE; LIU, 1981). The RCM-GL heuristic is likely faster than the GPS heuristic. Additionally, the RCMM heuristic (RCM-GL heuristic implemented in MATLAB (2015)) surpassed the GPS heuristic in bandwidth reduction in tests conducted by Esposito et al. (1998).

• Results of the heuristic of Burgess and Lai (1986) were compared with results of the CM and GPS heuristics and the heuristics of Konishi, Shiraishi and Taniguchi (1976), of Grooms (1972), of Akhras and Dhatt (1976), of Martin and Alarcon (1982), and of Puttonen (1983). As a outcome for these tests, the best results in bandwidth reduction were obtained by the heuristic of Burgess and Lai (1986). Although such tests were carried out in small instances and the authors did not compare the computational costs of the heuristics that were tested, their heuristic probably has a low computational cost because it is based on the CM and GPS heuristics.

• The WBRA heuristic showed better results in bandwidth reduction than the RCMM heuristic (RCM-GL heuristic implemented in MATLAB (2015)), than the GPS heuristic, and also than the heuristic of Esposito and Tarricone (1996). Results of the WBRA heuristic, presented by Esposito et al. (1998), outperformed results of the GPS heuristic in bandwidth reduction by 38% to 52%. In addition of achieving slightly faster results than the GPS heuristic in small instances (as shown by Esposito et al. (1998)), it should be noted that in an instance with 1005 vertices, the WBRA heuristic was approximately four times faster than the GPS heuristic.

• Results of the GGPS heuristic were on average 8,91% and 37,63% better than results of the GPS heuristic in bandwidth reduction in instances from two-dimensional and three-dimensional simulations, respectively, in tests conducted by Wang, Guo and Shi (2009). In some instances, the GGPS heuristic obtained results that outperformed results of the GPS heuristic by 90% in bandwidth reduction. Moreover, the authors stated that the GGPS heuristic achieved computational costs similar to the GPS heuristic. More specifically, in tests conducted by Wang, Guo and Shi (2009), the ratio of the average execution time of the GPS heuristic by the average execution time of the GGPS heuristic was 1.04.

5. Conclusions

An analysis of comparisons made by researchers among heuristics for bandwidth reduction of symmetric matrices was the main objective of this present study. The aim of this study was on comparisons made by researchers among different heuristics for bandwidth reduction of symmetric matrices. The 44 heuristics included in this review are those methods that are not based on metaheuristics. Possibly, other heuristics not based on metaheuristics exist; however, it is highly probable that results of the main ones were analyzed in this systematic review.

When taking into account bandwidth reduction of symmetric matrices for reduction of the computational cost of solving linear systems by iterative methods, as the Conjugate Gradient Method, the total cost of the resolution of the linear system may be higher when using a heuristic that reduces very much the bandwidth because, in most cases, high computational cost is demanded to perform the task. Thus, a heuristic not based on metaheuristic that reduces reasonably the bandwidth at a very low computational cost (i.e. milliseconds or less) may be better to solve the task than a metaheuristic-based heuristic that presents large bandwidth reduction at a high computational cost (i.e in seconds).

Furthermore, designed as heuristics, these algorithms are inherently dependent on the instances. In spite of this, comparisons and published results of the authors were analyzed and considered as correct. As a result of this analysis, 4 heuristics that are not based on metaheuristics were identified as presenting a reasonable bandwidth reduction at very low computational cost: RCM-GL, of 1981; Burgess-Lai, of 1986; WBRA, of 1998; and GGPS, of 2009.

In future studies, these 4 heuristics shall be implemented in order to compare their results and computational costs. Although heuristics for bandwidth reduction may be dependent on instances, the best heuristic(s) for bandwidth reduction at a very low computational cost is/are expected to be found. More specifically, it is expected that the heuristic(s) for bandwidth reduction of symmetric matrices with the best benefits (i.e. bandwidth reduction) per computational costs can be discovered, leading to the highest total computational cost reduction in the resolution of linear systems by the Conjugate Gradient Method.

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References

Akhras, G. and Dhatt, G. (1976), An automatic node relabelling scheme for minimizing a matrix of network bandwidth. *INT. J. NUMER. METH. ENG.*, 10(4), 787-797.

Akyuz, F. A. and Utku, S. (1968), An automatic relabelling scheme for bandwidth minimization of stiffness matrices. *American Institute of Aeronautics and Astronautics*, 6, 728-730.

Alway, G. G. and Martin, D. W. (1965), An algorithm for reducing the bandwidth of matrix of symmetrical configuration. *COMPUT. J.*, 8(3), 264-272.

Arany, I., Smyth F. and Szoda, L. (1971), An improved for reducing the bandwidth of sparse symmetric matrices. *In Proc. Int. Fed. for Inf. Processing*, 1246-1260, Amsterdam, Netherlands.

Arathi, P., Doss, L. J. T. and Kanakadurga, K. (2012), A constructive bandwidth reduction algorithm. *International Journal of Operational Research*, 15(3), 308-320.

Armstrong, B. A. (1984), A hybrid algorithm for reducing matrix bandwidth. *INT. J. NUMER. METH. ENG.*, 20, 1929-1940.

Barnard S., Photen, A. and Simon, H. (1995), A spectral algorithm for envelope reduction of sparse matrices. *Numerical Linear Algebra with Aplications*, 3, 317-334.

Benzi, M. (2002), Preconditioning techniques for large linear systems: a survey. *Journal of Computational Physics*, 182, 418-477.

Benzi, M., Szyld, D. B. and Duin, A. V. (1999), Orderings for incomplete factorization preconditioning of nonsymmetric problems. *SIAM J. Sci. Comput.*, 20(5), 1652-1670.

Bernardes, J. A. B. and Gonzaga de Oliveira, S. L. (2015), A systematic review of heuristics for profile reduction of symmetric matrices (to appear). *In International Conference on Computational Science (ICCS)*, Procedia Computer Science, Reykjavk, Iceland.

Boutora, Y., Takorabet, N., Ibtiouen, R. and Mezani, S. (2007), A new method for minimizing the bandwidth and profile of square matrices for triangular finite elements mesh. *IEEE Transactions on Magnetics*, 43(4), 1513-1516.

Burgess, D. A. and Giles, M. B. (1997), Renumbering unstructured grids to improve the performance of codes on hierarchical memory machines. *ADV. ENG. SOFTW.*, 28, 189-201.

Burgess, I. W. and Lai, P. K. F. (1986), A new node renumbering algorithm for bandwidth reduction. *INT. J. NUMER. METH. ENG.*, 23, 1693-1704.

Chagas, G. O. and Gonzaga de Oliveira, S. L. (2015), Metaheuristic-based heuristics for symmetric-matrix bandwidth reduction: a systematic review (to appear). *In International Conference on Computational Science (ICCS)*, Procedia Computer Science, Reykjavk, Iceland.

Cheng, K. (1973), Minimizing the bandwidth of sparse symmetric matrices. *Computing*, 11, 103-110.

Collins, R. J. (1976), Bandwidth reducing by automatic renumbering. *INT. J. NUMER. METH. ENG.*, 6, 345-356.

Crane, H. L. J., Gibbs, N. E., Poole, W. G. J. and Stockmeyer, P. K. (1976), Algorithm 508: Matrix bandwidth and profile reduction [F1]. *ACM T. MATH. SOFTWARE*, 2(4), 375-377.

Cuthill, E. H. (1972), Several strategies for reducing the bandwidth of matrices. *In Proc. of a Symp. on Sparse Matrices and Their App.*. The IBM Res. Symp. Series, New York, NY, 157-172.

Cuthill, E. H. and McKee, J. (1969), Reducing the bandwidth of sparse symmetric matrices. *In ACM '69 Proceedings of the 24th national conference*. New York, NY.

Das, R., Mavriplis, D. J., Saltz, J. H., Gupta, S. K. and Ponnusamy, R. (1992), The design and implementation of a parallel unstructured euler solver using software primitives. Tech. Rep. AD-A249 437, Inst. for Computer Applications in Science and Eng. - NASA, Virginia, VA.

Del Corso, G. M. and Romani, F. (2001), Heuristic spectral techniques for the reduction of bandwidth and work-bound of sparse matrices. *Numerical Algorithms*, 28, 117-136.

Doss, L. J. T. and Arathi, P. (2011), A bandwidth reduction algorithm for L-shaped and Z-shaped grid structured graphs. *Operations Research Letters*, 39, 441-446.

Dueck, G. W. and Jeffs, J. (1995), A heuristic bandwidth reduction algorithm. *Journal of Combinatorial Mathematics and Combinatorial Computing*, 18, 97-108.

Duff, I. S. and Meurant, G. A. (1989), The effect of ordering on preconditioned conjugate gradients. *BIT Numerical Mathematics*, 29(4), 635-657.

Esposito, A., Catalano, M. S. F., Malucelli, F. and Tarricone, L. (1998), A new matrix bandwidth reduction algorithm. *Operations Research Letters*, 23, 99-107.

Esposito, A. and Tarricone, L. (1996), Matrix bandwidth reduction with Tabu search: parallel implementation on Cray T3D and applications to microwave circuit design. *In Science and Supercomputing at CINECA: Report*, Voli, E. editor, Bologna, Italy, 440-450.
Everstine, G. C. (1979), A comparison of three resequencing algorithms for the reduction of matrix profile and wavefront. *INT. J. NUMER. METH. ENG.*, 14, 837-853.
Garey, M. R., Graham, R. L., Jonhson, D. S. and Knuth, D. E. (1978), Complexity results for bandwidth minimization. *SIAM Journal for Applied Mathematics*, 34, 477-495.
George, A. (1971), Computer implementation of the finite element method. *PhD thesis*, Stanford University, Stanford, CA.
George, A. and Liu, J. W. H. (1979), An implementation of a pseudoperipheral node finder. *ACM T. MATH. SOFTWARE*, 5(3), 284-295.
George, A. and Liu, J. W. H. *Computer solution of large sparse positive definite systems*, Prentice-Hall, Englewood Cliffs, 1981.
Gibbs, N. E. (1976), Algorithm 509: A hybrid profile reduction algorithm [F1]. *ACM T. MATH. SOFTWARE*, 2(4), 378-387.
Gibbs, N. E. A survey of bandwidth and profile reduction algorithms. *Tech. Rep. ADA090156*, Coll. William and Mary Williamsburg VA Dept. of Math. and Comp. Sc., Wiiliamsburg, VA, 1980.
Gibbs, N. E., Poole, W. G. and Stockmeyer, P. K. (1976), An algorithm for reducing the bandwidth and profile of a sparse matrix. *SIAM Journal on Numerical Analysis*, 13(2), 236-250.
Gibbs, N. E., Poole, W. G. and Stockmeyer, P. K. (1976), Comparison of several bandwidth and profile reduction algorithms. *ACM T. MATH. SOFTWARE*, 4(1), 322-330.
Glüge, R. (2010), Bandwidth reduction on sparse matrices by introducing new variables. *Revista Chilena de Ingeniería*, 18(3), 395-400.
GowriSankaran, C., Miller, Z. and Opatrny, J. (1990), A new bandwidth reduction for trees. *In Congressus Numerantium, Southeastern Conference on Combinatorics, Graph Theory and Computing*. 72, Pennsylvania, PA, 33-50.
GowriSankaran, C. and Opatrny, J. (1990), New bandwidth reduction algorithms. *In Cong. Num., Southeastern Conf. Combinatorics, Graph Theory and Comp.*, 76, Pennsylvania, PA, 77-88.
Grooms, H. R. (1972), Algorithm for matrix bandwidth reduction. *J. of the Struc. Div.*, 203-214.
Hestenes, M. R. and Stiefel, E. (1952), Methods of conjugate gradients for solving linear systems. *Journal of Research of the National Bureau of Standards*, 49(6), 409-436.
Kaveh, A. (1986), Ordering for bandwidth reduction. *Computer and Structures*, 24(3), 413-420.
Kaveh, A., *Structural Mechanics: Graph and Matrix Methods*, Research Studies Press LTD, Baldock, England, 2004.
Kaveh, A. and Roosta, G. R. (1999), A graph theorical method for frontwidth reduction. *ADV. ENG. SOFTW.*, 30(9-11), 789-797.
Kaveh, A. and Sharafi, P. (2012), Ordering for bandwidth and profile minimization problems via charged system search algorithm. *IJST Transactions of Civil Engineering*, 36, 39-52.
King, I. P. (1970), An automatic reordering scheme for simultaneous equations derived from network systems. *INT. J. NUMER. METH. ENG.*, 2(4), 523-533.
Konishi, I., Shiraishi, N. and Taniguchi, T. (1976), Reducing the bandwidth of structural stiffness matrices. *Journal of Structural Mechanics*, 4(2), 197-226.
Koohestani, B. and Poli, R. (2011), A hyper-heuristic approach to evolving algorithms for bandwidth reduction based on genetic programming. *In Research and Development in Intelligent Systems XXVIII*, 93-106, London, UK.
Lai, Y.-L. and Chang, G. (2004), On the profile of the corona of two graphs. *Information Processing Letters*, 89(6), 287-292.
Lanczos, C. (1952), Solutions of systems of linear equations by minimized iterations. *Journal of Research of the National Bureau of Standards*, 49(1), 33-53.
Levy, R. (1971), Resequencing of the structural stiffness matrix to improve computational efficiency. *Jet Propulsion Laboratory Quarterly Technical Review*, 1(2), 61-70.
Lewis, J. G. (1982), Implementations of the Gibbs-Poole-Stockmeyer algorithms and Gibbs-King algorithms. *ACM T. MATH. SOFTWARE*, 8, 180-189.
Lim, A., Lin, J. Rodrigues, B. and Xiao, F. (2006), Ant colony optimization with hill climbing for the bandwidth minimization problem. *Applied Soft Computing*, 6(2), 180-188.
Lim, A., Lin, J. and Xiao, F. (2007), Particle Swarm optimization and hill climbing for the bandwidth minimization problem. *Applied Intelligence*, 3(26), 175-182.

Lim, A., Rodrigues, B. and Xiao, F. (2006), Heuristics for matrix bandwidth reduction. *European Journal of Operational Research*, 69-91.
Livesley, R. K. (1960), The analysis of large structural systems. *COMPUT. J.*, 3(1), 34-39.
Luo, J. C. (1992), Algorithms for reducing the bandwidth and profile of a sparse matrix. *Computers and Structure*, 44, 535-548.
Mamaghani, A. S. and Meybodi, M. R. (2011), A learning automaton based approach to solve the graph bandwidth minimization problem. *In Proceedings of the 5th International Conference on Application of Information and Communication Technologies*, 1-5, IEEE.
Martí, R., Laguna, M., Glover, F. and Campos, V. (2001), Reducing the bandwidth of a sparse matrix with tabu search. *European Journal of Operational Research*, 135(2), 450-459.
Martin, A. F. and Alarcon, E. (1982), A contribution to the optimal ordering of tree structures. *Computer and Structures*, 15(3), 283-290.
Melosh R. J. and Bandford, R. M. (1969), Efficient solution of load-defletion equations. *J. of the Struc. Div.*, 95, 661-676.
Papadimitriou, C. H. (1976), The NP-completeness of bandwidth minimization problem. *Computing*, 16, 263-270.
Puttonen, J. (1983), Simple and effective bandwidth reduction algorithm. *INT. J. NUMER. METH. ENG.*, 19, 1139-1152.
Ramalho, M. A. and Corrêa, M. R. S. (1991), Minimização de banda em sistemas computacionais baseados no método dos elementos finitos. *Mecánica Computacional*, 12(8), 541-550.
Razzaque, A. (1980), Automatic reduction of frontwidth for finite element analysis. *INT. J. NUMER. METH. ENG.*, 15, 1315-1324.
Recuero, A. and Gutierrez, J. P. (1984), Effective bandwidth reduction algorithm for micro-computers. *In Proc. of the 1st Int. Conf. of Eng. Softw. for Microcomputers*, 1, Venice, Italy.
Rodrigues, J. S. (1975), Node numbering optimization in structural analysis. *American Society of Civil Engineers*, 101, 361-376.
Rosen, R. (1968), Matrix bandwidth minimization. *In Proceedings of the 23rd ACM/CSC-ER National Conference*, 585-595, Princeton, Brandon Systems.
Saad, Y. and Schultz, M. H. (1986), Gmres: A generalized minimal residual algorithm for solving nonsymmetric linear systems. *SIAM Journal on Scientific and Statistical Computing*, 7, 856-869.
Santos, M. I. G. and Groehs, A. G. (1988), Minimização de banda e perfil de matrizes de rigidez de estruturas. *Mecánica Computacional*, 8(4), 399-413.
Scott, D. G. and Han, R. P. S. (1994), Basis of an improved hybrid node renumbering algorithm for matrix bandwidth reduction. *COMPUT. METHOD. APPL. M.*, 118, 309-318.
Serra, J. L. F. d. A. (1997), Algoritmo para redução da banda de matrizes simétricas esparsas. *Mecánica Computacional*, 18(10), 567-576.
Smyth, W. F. (1985), Algorithms for the reduction of matrix bandwidth and profile. *Journal of Computational and Applied Mathematics*, 12, 551-561.
Smyth, W. F. and Arany, I. (1976), Another algorithm for reducing bandwidth and profile of a sparse matrix. *In Proceedings of U.S. National Computer Conference*, 987-994, New York, NY.
Snay, R. A. (1976), Reducing the profile of sparse symmetric matrices. *B. GEOD.*, 50, 341-352.
Taniguchi, T. and Shiraishi, N. (1980), New renumbering algorithm for minimizing the bandwidth of sparse matrices. *ADV. ENG. SOFTW.*, 2(4), 341-352.
Tarjan, R. E. Graph theory and gaussian elimination. *Technical Report STAN-CS-75-526*, Stanford University, Department of Computer Science, Stanford, CA, 1975.
Teixeira, F. G. and Groehs, A. G. (1991), Sistema de reordenação nodal para soluções do tipo banda. *IN XXV Jornadas sul-americanas de engenharia estrutural*, 10, Porto Alegre, Brasil, Associação Sul Americana de Engenharia Estrutural.
The MathWorks Inc. MATLAB, 1994-2015. *http://www.mathworks.com/products/matlab/index.html* - Access: May 1st, 2015.
Wang, P. T. R. Bandwidth minimization, reducibility, decomposition and triangularization of sparse matrices. *PhD thesis*, Comp. and Information Sc. Res. Center, Ohio State University, 1973.
Wang, Q., Guo, Y. C. and Shi, X. W. (2009), A generalized GPS algorithm for reducing the bandwidth and profile of a sparse matrix. *In Progress In Electromagnetics Research*, 90, 121-136, EMW Publishing, Cambridge, MA.