# A BENDERS STOCHASTIC DECOMPOSITION APPROACH FOR THE OPTIMIZATION UNDER UNCERTAINTY OF INVENTORY CONTROL CONSIDERING PERIODIC REVIEW POLICY

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## ABSTRACT
This paper presents the application of a decomposition scheme for a problem of inventory control of one echelon, one item, uncertain demand and based on a (R, S) replacement policy using two-stage stochastic programming, as originally proposed by Cunha et al. (2014). We proposed two methodologies, one based on L-shaped (Van Slyke & Wets, 1969), and the other, based on multi-cut L-shaped (Birge & Louveaux, 1988), for accelerating the computational solution process. The results showed that both proposed methodologies are capable to satisfactorily improve the solution process in terms of computational time. However, despite the fact that the single-cut method always required more iterations than the multi-cut version to obtain optimal solutions, the single-cut presented better performance in terms of computational time, especially when large number of scenarios and periods are considered.

Keywords: L-shaped method, two-stage stochastic programming, inventory control.

## 1. INTRODUCTION

Inventories are important in all types of organizations because they provide means to protect the business from uncertainties inherent to the business (such as demand fluctuations, unexpected lead times delays, and so forth). However, having inventories profoundly affects daily operations, since they must be counted, paid, and used in operations to meet customers and administrators' demands (Krajewski et al., 2009). The key questions that inventory management aims to answer, usually subject to a variety of circumstances are: when ordering, how much to order and how much to keep as safety stocks (Namit & Chen, 1999; Silva, 2009).

The major issue with regard to inventory management is to ensure the availability of product to the final customer at the lowest cost. In academic literature, there are several proposals approaching inventory control policies associated with mathematical models that aim at minimization of costs related to stock. These models can be divided into two groups: deterministic models, in which it is assumed that all parameters are previously known, and probabilistic models, in which one or more parameters, such as demand and lead time, are modeled as stochastic.

Regarding deterministic models for inventory planning, one of the best-known models is the EOQ (Economic Order Quantity) model, developed by Harris (1913). This model represents the basis for other classic models, such as the EPQ (Economic Production Quantity), in which the assumption of instantaneous replenishment is replaced by the assumption that the replenishment order is received at constant finite rate over time. According to Pentico & Drake (2009), despite the EOQ model being subject to criticism due to its mathematical simplifications, researches have shown that it is widely successful when used in practice. Over the years, new formulations have been proposed for deterministic models, with the objective of reducing their simplifications and making them more general (see, for example, Pentico & Drake (2009), Montgomery et al. (1973), Park (1982) and Pentico et al. (2009)).

In the context of inventory management, it is well known that optimal policies can be obtained by using dynamic programming through the Wagner-Whitin algorithm whenever the demand is considered deterministic (Axsater, 2006). However, the simplifying considerations adopted in deterministic models, especially the assumption of prior knowledge of demand behavior, do not represent the practical reality, which ultimately motivates the development of inventory management models that are capable to take uncertainty into consideration.

Among the classical systems of inventory control in inventory management literature, which are used when considering the uncertainty in demand, we mention the following: (R, Q), (R, S), (R, s, S), (s, S) e (s, Q). In these systems R, Q, s and S means periodic review periods, reorder quantities, reorder points, and inventory level targets, respectively. In systems (R, Q), (R, S), and (R, s, S), in every R units of time (periodic review), a fixed quantity Q of the item is ordered in the first, while in the others, a sufficient variable amount to raise inventory position to level S is requested in each inventory review, and the third an order is performed only if the stock position is less than or equal to s. The systems (s, S) and (S, Q) assume continuous review, where a quantity is ordered when the position of stock is less than or equal to the reorder points, being that in the (s, S) a variable quantity is ordered, sufficiently to raise the stock level position to target level S while in the second, a fixed amount Q is ordered.

In real applications, it is more frequent that stocks are checked periodically rather than in a continuous fashion (Fattahi et al., 2014). This is due to the advantages obtained in periodic review. The periodic review easily reveals the amount of work involved and generally is cheaper than continuous review, which requires a real-time information system. According to Hadley & Whitin (1963) the periodical review in the inventory replenishment policy is widely used because it requires less transactional effort, allowing ease of planning for calculating workload requirements, facilitates attending both customers and suppliers needs, allowing a better replacement coordination, especially when you have multiple items, and generates greater stability to the system.

Several models considering uncertain demand were proposed by Hadley & Whitin (1963) and Silver & Peterson (1998), and, in a most of them, cost parameters were considered fixed throughout the planning horizon. The stochastic demands were approximated to known probability distribution models, (both discrete or continuous distribution models, depending on the size of the problem in terms of demand and ordering quantities). In the model proposed by Hadley & Whitin (1963), for example, the main constraint is that the demand for each period is time-wise independent and normally distributed, which is a strong hypothesis, given that in the real-world demand, and other parameters, may depend on factors such as the uncertain market conditions, cost and the time of the year (seasonality).

One way to relax the hypothesis of having to model demand stochastic behavior in a rather simplistic fashion is to use two-stage stochastic optimization models (Shapiro & Philpott, 2007). This model is compatible with the aforementioned inventory policies, since it can be used to model control variables R, S, and Q in each system as first-stage variables, which are those that represent decisions that should be made prior to knowing how the uncertainty unveils. The remaining variables, so-called second-stage or recourse variables, which are linked to control decisions, are determined after knowing how the uncertainty unveils. One of the main advantages of the two-stage stochastic programming framework is that the stochastic parameters can be modeled without assuming any restrictive hypothesis on the stochastic phenomenon, provided that their behavior can be approximated by a discrete set of possible scenarios associated with their respective probabilities, which allows a closer representation of the real demand behavior of a particular item.

The use of two-stage stochastic programming for inventory control can be seen, for example, in Fattahi et al. (2014), where they model a two-echelon network based on the continuous review policy (s, S), considering a single item and the demand parameter as uncertain. Cunha et al. (2014) modeled a one-echelon network, considering a single item and the demand as uncertain parameter. In this case, it was adopted the periodic review policy (R, S).

The model developed by Cunha et al. (2014) was based on mixed-integer nonlinear programming (MINLP), further linearized into a mixed-integer linear program (MILP) through exact reformulations, at the cost of increasing the number of variables and constraints in the model. The increase of variables and constraints added to the fact that the aforementioned problem is a MILP makes it challenging to solve its deterministic equivalent version due to computational the computational burden.

In Cunha et al. (2014), the results were compared with those of the model of Hadley & Whitin (1963) and it was observed that increasing the number of periods and scenarios would lead to a reduction in the absolute percentage error of the minimum cost. Thus, for best results, it is made necessary to perform simulations considering large numbers of scenarios and periods, which makes the computational performance of full-space deterministic equivalent problem intractable. Hence, it becomes necessary to use a method to accelerate the computational solution process, in particular effective methods for solving large-scale problems based on decomposition.

Slyke & Wets (1969) presented the first work using Benders decomposition (Benders, 1962) in two-stage stochastic programming, which is often referred as the L-shaped method. Exploiting the two-stage stochastic problem structure, Birge & Louveaux (1988) extended the L-shaped method to a multi-cut version (multi-cut L-shaped). The computational efficiency of these methods is widely proved in literature, especially in the context of two-stage stochastic optimization problems (see Castro et al. (2009), Khodr et al. (2009), Alysson et al. (2012), Bertsimas et al. (2013), and Oliveira et al. (2014), for example)

In this context, this paper aims at developing a solution approach based on Benders decomposition for the model proposed in Cunha et al. (2014) when considering large-scale problems (i.e., with a large number of scenarios). Moreover, we compare the performance of two different versions of the algorithm, namely the traditional L-shaped method the multi-cut version presented by Birge and Louveaux (1998) for this specific problem.

The paper is organized as follows: Section 2 describes the proposed mathematical model in Cunha et al. (2014). Section 3 presents an algorithm based in traditional L-shaped decomposition, while Section 4 presents the multi-cut framework. Section 5 describes how the computational experiments were performed and reports the numerical results obtained. Section 6 draws some conclusion and future works.

## 2. MATHEMATICAL MODEL

The following notation is used to present the mathematical model proposed by Cunha et al. (2014). For the sake of notation simplicity, the domains of summations will be omitted, except when the summation is evaluated only in a subset of the natural domain. When there is no mention of this fact, its domain should be considered as original set to which the index refers.

**Sets**

$P$ – Time periods
$\Omega$ – Scenarios
$\tau$ – Review periods

**Indexes**

$p \in P$ – Time period
$\xi \in \Omega$ – Scenario
$r \in \tau$ – Review period

**Parameters**

$B^p$ – Stock-out cost per unit of item in the period $p$
$CF^p$ – Fixed cost of ordering in the period $p$

$D(\xi)^p$ – Demand for the item in scenario $\xi$ and period $p$
$H^p$ – Inventory cost per unit of item in the period $p$
$\overline{ITI}$ – Upper bound limit for the total position of the item inventory
$\overline{S}$ – Upper bound for the maximum level of the item inventory
$Pr(\xi)$ – Probability of scenario $\xi$
$W_p^r$ – Auxiliary parameter that indicates the period in which an ordering occurs depending on the value $r$; $w_p^r \in \{0,1\}$; $r = 1, \dots, NR$; $p = 1, \dots, NP$

**Variables**

$a(\xi)^p$ – Amount of met demand in scenario $\xi$ and period $p$
$f(\xi)^p$ – Amount of unmet demand in scenario $\xi$ and period $p$
$i(\xi)^p$ – On-hand inventory in scenario $\xi$ and period $p$
$it(\xi)^p$ – Position of total inventory (on-hand inventory plus pending ordering) in scenario $\xi$ and at the end of period $p$
$iti(\xi)^p$ – Position of total inventory (on-hand inventory plus pending ordering) in scenario $\xi$ and at the beginning of the period $p$
$itiv(\xi)^p$ – Auxiliary variable for position of total inventory (on-hand inventory plus pending ordering) in scenario $\xi$ and at the beginning of the period $p$
$q(\xi)^p$ – Order quantity of the item in scenario $\xi$ and period $p$
$s$ – Order-up-to levels of the inventory item over the time horizon (S)
$sv^p$ – Auxiliary variable for order-up-to levels of the inventory item in period $p$
$v^p$ – Indicates whether an order of the item in period $p$ exist or not; $v^p \in \{0,1\}$
$u^r$ – Auxiliary variable in determining the size of cycle $R$; $u^r \in \{0,1\}$.

The complete deterministic equivalent formulation of the two-stage stochastic model presented by Cunha et al. (2014) can be stated as follows:

$$\min_{a,f,i,it,iti,itiv,q,s,sv,v,u} \sum_p CF^p\, v^p + \sum_{\xi,p} Pr(\xi)\, [H^p\, i(\xi)^p + B^p f(\xi)^p] \qquad (1)$$

Subject to:

$$\sum_r u^r = 1 \qquad (2)$$
$$\sum_r W_p^r\, u^r = v^p \quad \forall p \qquad (3)$$
$$0 \le s \le \overline{S} \qquad (4)$$
$$i(\xi)^{p-1} + q(\xi)^{p-TE} = i(\xi)^p + a(\xi)^p \quad \forall p \qquad (5)$$
$$it(\xi)^{p-1} + q(\xi)^p = it(\xi)^p + a(\xi)^p \quad \forall p \qquad (6)$$
$$a(\xi)^p + f(\xi)^p = D(\xi)^p \quad \forall p \qquad (7)$$
$$q(\xi)^p = sv^p - itiv(\xi)^p \quad \forall p \qquad (8)$$
$$sv^p \le \bar{S} v^p \quad \forall p \qquad (9)$$
$$sv^p \le s \quad \forall p \qquad (10)$$
$$sv^p \ge s - \bar{S}(1 - v^p) \quad \forall p \qquad (11)$$
$$itiv(\xi)^p \le \overline{ITI} v^p \quad \forall p \qquad (12)$$
$$itiv(\xi)^p \le iti(\xi)^p \quad \forall p \qquad (13)$$
$$itiv(\xi)^p \ge iti(\xi)^p - \overline{ITI}(1 - v^p) \quad \forall p \qquad (14)$$
$$iti(\xi)^p = it(\xi)^{p-1} \quad \forall p \qquad (15)$$
$$u^r \in \{0,1\} \quad \forall r \qquad (16)$$
$$v^p \in \{0,1\} \quad \forall p \qquad (17)$$
$$sv^p \ge 0 \quad \forall p \qquad (18)$$
$$a(\xi)^p, i(\xi)^p, it(\xi)^p, iti(\xi)^p, itiv(\xi)^p, q(\xi)^p \ge 0 \quad \forall p \qquad (19)$$

Expression (1) models the total cost that one seeks to minimize. The first term refers to costs of ordering over the planning horizon considered. The second term refers to inventory costs and stock-out costs over the planning horizon. Constraint (2) indicates that there is exactly one value that determines the size of cycle $R$ ($R = r,$ when $u^r = 1$). Constraint (3) indicates that orders depend on the choice of R and that the first order always occurs in the first period of the planning horizon (according to the definition of values $w_p^r$). Constraint (4) determines the lower and upper bounds for the order-up-to level variable. Constraint (5) represents the balance of stocks in hand from one period to the next, in each scenario. Constraint (6) represents the balance of the position of total inventory (on-hand plus orders in transit) of an item from one period to the next, in each scenario $\xi$. Constraint (7) represents the met and unmet demand in each period, for each scenario $\xi$. The Constraints (8)-(15) represent the exact linearization of the following equation:

$$q(\xi)^p = (s - it(\xi)^{p-1})\, v^p \qquad \forall p \qquad (20)$$

Constraint (20) represents the amount that must be ordered in every R periods over the planning horizon for each scenario $\xi$. It is a nonlinear constraint, which makes the model a MINLP problem, but the exact linearization of this equation allows the proposed model to be stated as a mixed-integer linear programming (MILP) problem, which is more amenable in terms of computational complexity. At last, (16)-(19) present domains of the decision variables.

## 3. STOCHASTIC BENDERS DECOMPOSITION (L-SHAPED METHOD)

The model proposed in the previous section can be defined as an optimization model with binary and continuous first-stage variables, composed by equations (2-4), (9-11), (16-18) and the first term of objective function, and continuous second-stage variables, composed by equations (5-8), (12-15), (19) and the second term of objective function. Moreover, the model has relatively complete recourse (Birge & Louveaux, 1997) that is, for any feasible first stage solution, the second stage problem is feasible. This occurs, because every time that some part of the demand is not fulfilled, a penalty is imposed without precluding the existence of an optimal solution for the second-stage problem. Such characteristics allow us to develop a decomposition framework based on Benders decomposition (Benders, 1962) applied to stochastic optimization. The stochastic two-stage structure allows modeling the master problem from the first-stage problem and the slave problem from the second-stage problem.

We start by stating the slave problem, which is the dual representation of the second-stage problem, where the complicate variables $sv^p$ and $v^p$ are considered as fixed parameters $\overline{sv}^p$ and $\bar{v}^p$, and $\alpha,\ \beta,\ \gamma,\ \sigma,\ \pi,\ \mu,\ \rho,\ \omega$ are the dual variables associated with constraints (5) to (8) and (12) to (15), respectively The slave problem can, thus, be stated as follows:

$$Q(\bar{v}, \overline{sv}) =$$

$$= \max_{\alpha,\beta,\gamma,\sigma,\pi,\mu,\rho,\omega} \sum_{\xi,p} Pr(\xi)[D(\xi)^p\,\gamma(\xi)^p + \overline{ITI}\bar{v}^p\pi(\xi)^{(p)} + \overline{ITI}\,(\bar{v}^p - 1)\rho(\xi)^p + \overline{sv}^p\sigma(\xi)^p] \qquad (21)$$

Subject to:

$$\beta(\xi)^{p+1} - \beta(\xi)^p - \omega(\xi)^{p+1} \le 0 \qquad \forall p \qquad (22)$$
$$\alpha(\xi)^{p+1} - \alpha(\xi)^p \le H^p \qquad \forall p \qquad (23)$$
$$-\alpha(\xi)^p - \beta(\xi)^p + \gamma(\xi)^p \le 0 \qquad \forall p \qquad (24)$$
$$\alpha(\xi)^{p+TE} + \beta(\xi)^p + \sigma(\xi)^p \le 0 \qquad \forall p \qquad (25)$$
$$\gamma(\xi)^p \le B^p \qquad \forall p \qquad (26)$$
$$\sigma(\xi)^p + \pi(\xi)^p + \mu(\xi)^p + \rho(\xi)^p \le 0 \qquad \forall p \qquad (27)$$
$$\mu(\xi)^p + \rho(\xi)^p + \omega(\xi)^p \le 0 \qquad \forall p \qquad (28)$$
$$\alpha(\xi)^p, \beta(\xi)^p, \gamma(\xi)^p, \sigma(\xi)^p, \omega(\xi)^p \quad \in \mathbb{R}^{p\times\xi} \qquad \forall p \qquad (29)$$

$$\rho(\xi)^p \geq 0 \quad \forall p \quad (30)$$
$$\pi(\xi)^p, \mu(\xi)^p \leq 0 \quad \forall p \quad (31)$$

The slave problem is easier to solve in comparison to the original problem because it has less variables and constraints and it is continuous. Notice that it also can be decomposed and solved for each scenario independently, which might leads to improved solution times, especially when the number of scenarios is large.

The formulation of the master problem is given by:

$$\min_{v,u,s,sv} \sum_p CF^p\, v^p + m \quad (32)$$

Subject to:

$$\sum_r u^r = 1 \quad \forall p \quad (33)$$
$$\sum_r W_p^r\, u^r = v^p \quad (34)$$
$$0 \leq s \leq \bar{S} \quad \forall p \quad (35)$$
$$sv^p \leq \bar{S} v^p \quad \forall p \quad (36)$$
$$sv^p \leq s \quad \forall p \quad (37)$$
$$sv^p \geq s - \bar{S}(1 - v^p) \quad (38)$$
$$m \geq Q(v, sv) \quad (39)$$
$$u^r \in \{0,1\} \quad \forall r \quad (40)$$
$$v^p \in \{0,1\} \quad \forall p \quad (41)$$
$$sv^p, m \geq 0 \quad \forall p \quad (42)$$

Inequality (39) is not a constraint defined explicitly, but only implicitly, by a number of optimization problems. The main idea of L-shaped method (Slyke & Wets, 1969) is to relax constraint (39) and replace it by a number of cuts, which may be gradually added following an iterative solving process. These cuts, defined as supporting hyperplanes of the second-stage objective function, eventually provides a good estimation for the value of $Q(v, sv)$ in a finite number of iterations. In other words, it is important to observe that there is one constraint (39) for each extreme point of slave problem in the complete master problem. It is true that there may be an enormous number in a problem of even a moderate size. However, it is expected that only a small fraction of the constraints will be binding in the optimal solution.

Initially, the master problem is solved, providing initial $sv^p$ and $v^p$ and a convergence check is made (because of the definition of initial upper bound, the algorithm never stop in the first test). Next the slave problem is solved for $\alpha,\ \beta,\ \gamma,\ \sigma,\ \pi,\ \mu,\ \rho, \omega$ given initial $sv^p$ and $v^p$ determined by the master in initialization and the upper bound is updated. Then, a constraint (cut) involving $\alpha, \beta, \gamma, \sigma, \pi, \mu, \rho, \omega$ must be added to the master, which will provide new lower bound and $sv^p$ and $v^p$ solutions. If the convergence criterion is not satisfied, the slave is solved with the current $sv^p$ and $v^p$ solutions and the loop starts again. This process continues until optimality (within a tolerance level $\epsilon$) can be obtained. The proposed methodology is shown in Figure 1.

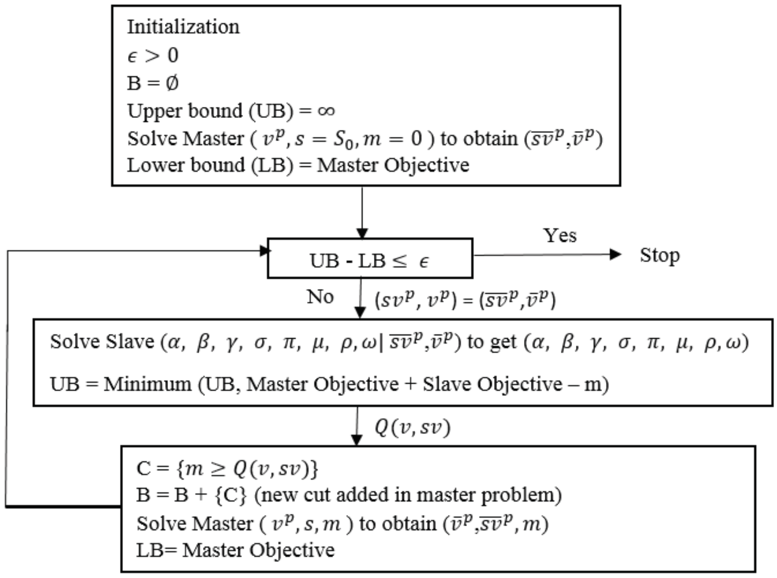


***Figure* 1** - *Flowchart of L-shaped method for proposed methodology*

**4. MULTICUT STOCHASTIC BENDERS DECOMPOSITION**

The structure of stochastic programs allows one to add multiple cuts to the master problem instead of one in each major iteration. Birge & Louveaux (1988) show that the use of such a framework may greatly speed up convergence. The main idea, is that in this case, instead of generating a single cut at each interaction, one can add multiple cuts to the master problem, equal to the number of scenarios considered and, therefore, fewer iterations might be necessary to achieve the optimal solution. Birge & Louveaux (1988) showed that the maximum number of iterations for the multi-cut procedure is given by

$$1 + |\Omega| (q^m - 1), \tag{43}$$

while the maximum number of iterations for the single-cut procedure is given by

$$[1 + |\Omega|(q^m - 1)]^m, \tag{44}$$

where $m$ represents the number of recourse constrains, $|\Omega|$ the number of the different realizations of $\xi$ and $q$ represents the total of slopes for the second-stage problem. Although $q$ might turn out to be complicated to calculate for real-world problems, (43) show that the maximum number of iterations, for multi-cut algorithm, needed for reaching the optimum grows linearly with the number of scenarios, while (44) shows that it grows exponentially for the traditional single-cut algorithm.

The main difference between the multi-cut L-shaped algorithms and its traditional version is the formulation of the master problem, which must be conveniently adequate to the multi-cut framework, as follows. The flowchart of multi-cut L-shaped method for proposed methodology is shown in Figure 2.

$$\min_{s,sv,u,v} \sum_p CF^p\, v^p + \sum_\xi Pr\ (\xi) m(\xi) \tag{45}$$

Subject to:

(33) to (38), (40) to (42)

$$m(\xi) \geq \sum_p [D(\xi)^p\, \gamma(\xi)^p - \overline{ITI}\, \rho(\xi)^p + \overline{ITI}(\pi(\xi)^{(p)} + \rho(\xi)^p) v^p + sv^p \sigma(\xi)^p] \quad \forall \xi \tag{46}$$

The framework of slave problem may be conveniently adequate too, as follows.

$$\max_{\alpha,\beta,\gamma,\sigma,\pi,\mu,\rho,\omega} \sum_{\xi,p} [D(\xi)^p\, \gamma(\xi)^p + \overline{ITI}\bar{v}^p \pi(\xi)^{(p)} + \overline{ITI}\,(\bar{v}^p-1)\rho(\xi)^p + \overline{sv}^p\, \sigma(\xi)^p] \quad (47)$$

Subject to:

(22) to (31)

## 5. COMPUTATIONAL RESULTS AND ANALYSIS

In this section, we present the numerical experiments we performed using the single-cut and multi-cut approaches in the MILP model for inventory control (R, S) proposed for Cunha et al. (2014) in order to analyze their performances. The proposed methodologies were implemented using AIMMS 3.14. The LP slave problems was successively solved within the decomposition framework using CPLEX 12.5. All experiments were performed on an Intel processor core I7 2.0 GHz with 8 Gb RAM. Demand scenarios were randomly generated, for each combination of scenarios, periods and periodicities, following a normal distribution with average $\mu = 50$ and variance $\sigma^2 = 75$. The tolerance level $\epsilon$ was set to 0.00001 in all cases, which, in practice, provides exact values for variables $R$ and $S$.

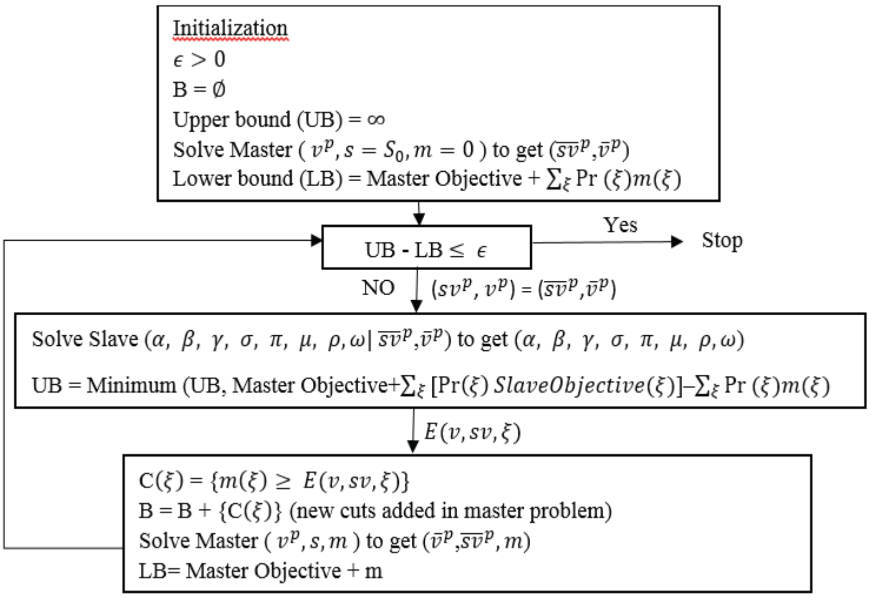


**Figure 2** - Flowchart of multi-cut L-shaped method for proposed methodology

Four different instances were considered in our experiments, in which each costs was independently and separately increased by 50% in respect to Instance 1, as can be seen in the Tables 1 to 4. For each instance, experiments were performed considering 10 and 20 periodicities, 36 and 72 periods and 50, 100, 250 and 500 scenarios. All presented results are from a single sample (for each sample size). For each combination of scenario, period, periodicities and instance considered, Tables 1 and 2 provide solution times, in seconds, for full-space version of problem (CPLEX), single-cut (SCut) and multi-cut (MCut) methods, while Tables 3 and 4 show the total number of iterations required for the algorithms based on the single-cut and multi-cut methods to reach the optimum solution within the defined tolerance level.

**Table 1 -** Solution times[s] for instances 1 and 2.

| Scenarios | Periods | Periodicities | Instance 1 | | | Instance 2 | | |
|---|---|---|---|---|---|---|---|---|
| | | | $B^p$=25; $CF^p$=25; $H^p$=0.2 | | | $B^p$=25; $CF^p$=37.5; $H^p$=0.2 | | |
| | | | CPLEX | SCut | MCut | CPLEX | SCut | MCut |
| 50 | 36 | 10 | 11.36 | 4.65 | 8.70 | 11.50 | 4.15 | 8.71 |
| | | 20 | 11.56 | 5.18 | 9.22 | 12.20 | 5.46 | 9.66 |
| | 72 | 10 | 48.21 | 9.63 | 21.86 | 48.69 | 9.68 | 20.61 |
| | | 20 | 50.56 | 10.75 | 22.49 | 51.17 | 10.12 | 22.49 |
| 100 | 36 | 10 | 31.44 | 7.38 | 22.01 | 33.43 | 8.15 | 21.67 |
| | | 20 | 43.31 | 9.95 | 23.41 | 41.32 | 10.83 | 21.51 |
| | 72 | 10 | 186.22 | 15.16 | 41.35 | 202.59 | 17.00 | 41.12 |
| | | 20 | 221.00 | 19.08 | 72.88 | 219.06 | 16.55 | 69.32 |
| 250 | 36 | 10 | 233.63 | 21.29 | 57.34 | 241.07 | 19.59 | 58.60 |
| | | 20 | 278.67 | 24.61 | 69.22 | 282.89 | 22.56 | 70.29 |
| | 72 | 10 | 1449,65 | 35.25 | 176.90 | 1456.32 | 37.10 | 175.39 |
| | | 20 | 1501.74 | 36.45 | 201.58 | 1553.41 | 35.45 | 193.82 |
| 500 | 36 | 10 | 1059.29 | 36.35 | 204.43 | 1072.23 | 36.59 | 200.50 |
| | | 20 | 1196.75 | 38.96 | 189.50 | 1227.53 | 53.33 | 170.40 |
| | 72 | 10 | 5309.28 | 66.72 | 459.37 | 5290.30 | 71.93 | 452.19 |
| | | 20 | 6490.64 | 66.52 | 540.73 | 6490.80 | 70.72 | 538.64 |

**Table 2 -** Solution times[s] for instances 3 and 4.

| Scenarios | Periods | Periodicities | Instance 3 | | | Instance 4 | | |
|---|---|---|---|---|---|---|---|---|
| | | | $B^p$=37.5; $CF^p$=25; $H^p$=0.2 | | | $B^p$=25; $CF^p$=25; $H^p$=0.3 | | |
| | | | CPLEX | SCut | MCut | CPLEX | SCut | MCut |
| 50 | 36 | 10 | 11.92 | 5.20 | 10.45 | 9.42 | 4.60 | 10.68 |
| | | 20 | 14.98 | 6.46 | 10.28 | 11.82 | 4.42 | 9.55 |
| | 72 | 10 | 54.22 | 10.83 | 32.06 | 43.59 | 7.19 | 15.37 |
| | | 20 | 62.51 | 10.78 | 31.27 | 47.67 | 9.52 | 25.84 |
| 100 | 36 | 10 | 38.10 | 9.23 | 19.13 | 30.02 | 6.52 | 22.12 |
| | | 20 | 48.96 | 10.48 | 28.32 | 32.40 | 9.13 | 25.61 |
| | 72 | 10 | 254.27 | 18.99 | 57.95 | 162.20 | 13.77 | 42.71 |
| | | 20 | 274.34 | 21.17 | 78.52 | 182.86 | 17.76 | 50.75 |
| 250 | 36 | 10 | 325.47 | 21.14 | 49.56 | 209.32 | 17.61 | 63.43 |
| | | 20 | 367.32 | 24.74 | 83.11 | 182.64 | 18.90 | 77.18 |
| | 72 | 10 | 1923.53 | 37.79 | 191.03 | 1071.18 | 29.38 | 146.80 |
| | | 20 | 2424.57 | 42.46 | 246.66 | 1030.86 | 32.29 | 202.18 |
| 500 | 36 | 10 | 1367.57 | 46.96 | 110.05 | 887.05 | 34.70 | 189.48 |
| | | 20 | 1986.46 | 52.42 | 221.76 | 1086.28 | 43.44 | 225.64 |
| | 72 | 10 | 7496.49 | 67.63 | 458.74 | 3920.60 | 62.41 | 375.77 |
| | | 20 | 12075.14 | 74.68 | 467.23 | 4003.32 | 68.54 | 580.73 |

**Table 3 -** Total number of iterations for instances 1 and 2.

| Scenarios | Periods | Periodicities | Instance 1 | | Instance 2 | |
|---|---|---|---|---|---|---|
| | | | $B^p$=25; $CF^p$=25; $H^p$=0.2 | | $B^p$=25; $CF^p$=37.5; $H^p$=0.3 | |
| | | | SCut | MCut | SCut | MCut |
| 50 | 36 | 10 | 32 | 20 | 30 | 20 |
| | | 20 | 32 | 22 | 32 | 22 |
| | 72 | 10 | 30 | 22 | 30 | 22 |
| | | 20 | 32 | 23 | 32 | 23 |
| 100 | 36 | 10 | 30 | 22 | 32 | 22 |
| | | 20 | 35 | 23 | 35 | 23 |
| | 72 | 10 | 33 | 22 | 32 | 22 |
| | | 20 | 34 | 26 | 33 | 26 |
| 250 | 36 | 10 | 32 | 22 | 32 | 22 |
| | | 20 | 37 | 23 | 34 | 23 |
| | 72 | 10 | 34 | 24 | 34 | 24 |
| | | 20 | 35 | 26 | 34 | 26 |
| 500 | 36 | 10 | 33 | 24 | 33 | 24 |
| | | 20 | 34 | 23 | 36 | 22 |
| | 72 | 10 | 35 | 24 | 35 | 24 |
| | | 20 | 35 | 26 | 36 | 26 |

**Table 4 -** Total number of iterations for instances 3 and 4.

| Scenarios | Periods | Periodicities | Instance 3 | | Instance 4 | |
|---|---|---|---|---|---|---|
| | | | $B^p$=37.5; $CF^p$=25; $H^p$=0.2 | | $B^p$=25; $CF^p$=25; $H^p$=0.3 | |
| | | | SCut | MCut | SCut | MCut |
| 50 | 36 | 10 | 35 | 23 | 29 | 22 |
| | | 20 | 36 | 23 | 29 | 22 |
| | 72 | 10 | 34 | 24 | 29 | 19 |
| | | 20 | 34 | 27 | 31 | 22 |
| 100 | 36 | 10 | 36 | 20 | 27 | 21 |
| | | 20 | 36 | 25 | 32 | 24 |
| | 72 | 10 | 37 | 24 | 31 | 21 |
| | | 20 | 37 | 27 | 32 | 23 |
| 250 | 36 | 10 | 38 | 21 | 29 | 22 |
| | | 20 | 38 | 25 | 32 | 24 |
| | 72 | 10 | 38 | 25 | 30 | 23 |
| | | 20 | 39 | 28 | 32 | 25 |
| 500 | 36 | 10 | 38 | 20 | 31 | 22 |
| | | 20 | 37 | 24 | 35 | 24 |
| | 72 | 10 | 36 | 23 | 33 | 23 |
| | | 20 | 39 | 25 | 35 | 27 |

Analyzing Tables 1 and 2, we notice that both the L-shaped (Slyke & Wets, 1969) and multi-cut L-shaped (Birge & Louveaux, 1988) methods were capable of improving the performance in terms of computational time when compared with the direct solution of the full-space version of the problem, especially when the number of scenarios and periods increase. Considering the case with larger scales (500 scenarios, 72 times and 20 periodicities), the full-space model presented an average solution time of 7265 seconds, while the proposed L-shaped and multi-cut L-shaped methods presented average computational times of 70 and 532 seconds, respectively.

As can be seen in Tables 3 and 4, the multi-cut version, in all cases, took less iterations than the single-cut version to reach the optimal solution. However, the single-cut approach always presented shorter solution times, as shown in the Tables 1 and 2. This occurs due to the fact that in multi-cut approach, more time is spent solving the master problem in each iteration. This is related with the fact that in the single-cut version, a single cut is added in the master problem in each iteration, while in the multi-cut version, the total of cuts added in each iterations is equal to the number of scenarios, significantly increasing the size, and therefore, their solution time. This can be verified, for example, when examining Tables 1, 2, 3 and 4 together. It seems that, when we increase the total of scenarios, periods or periodicities in each instance, the number of iterations required in both methods generally increases, but with little variation. Thus, increasing the amount of scenarios or periods causes a smaller impact in the computational time for the single-cut method when compared to the multi-cut version. This can also be seen in Figures 3, 4 and 5, where some data from previous tables were used to compare the performances of the proposed methodologies in terms of solution time, total quantity of iterations, and solution time per total of iterations, respectively. Table 1 also shows that the solution time for both methods is not relevantly affected by the variation of the number of periodicities and the set costs, when compared to the variation the total of periods or scenarios.

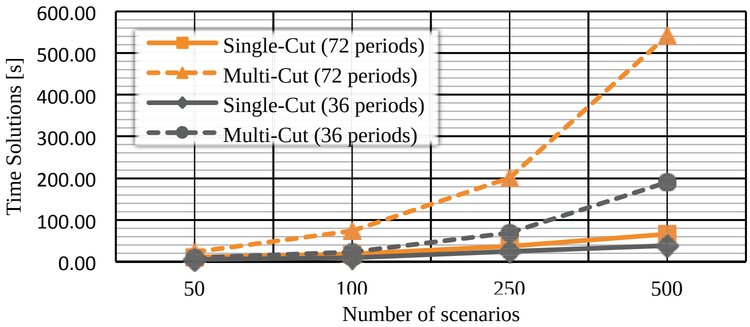


**Figure 3** - Comparison of computational times (20 periodicities and Instance 1).

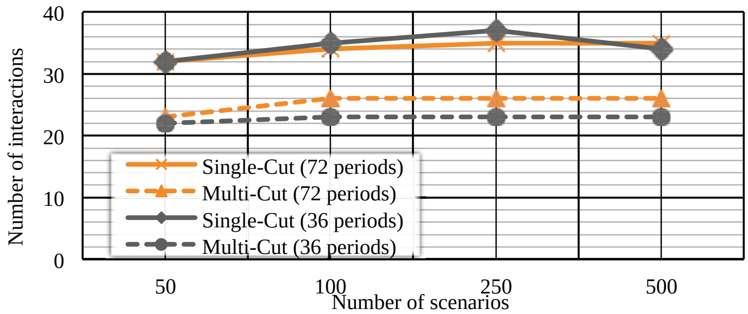


**Figure 4** - Comparison of number of iterations (20 periodicities and Instance 1).

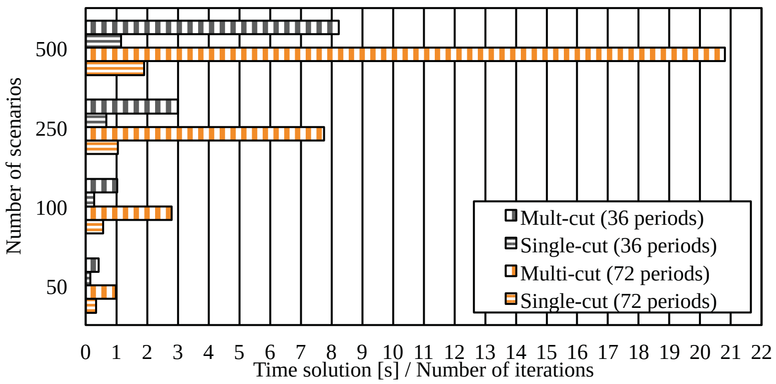


**Figure 5** - Comparison of time solution/number of iterations (20 periodicities and Instance 1).

## 6. CONCLUSIONS

In this paper, we proposed two approaches based on single-cut and multi-cut L-shaped methods for solving the optimal (R, S) inventory replenishment policy problem with periodic review by using a two-stage stochastic programming model, as proposed by Cunha *et al.* (2014).

The results showed that the proposed methodology was capable of improving the solution process in terms of computational time in a satisfactory manner. However, despite the single-cut version always required more iterations than the multi-cut version to obtain optimal solutions, the single-cut presented better performance in terms of solution time, especially larger numbers of scenarios and periods. Considering the case with larger scales (500 scenarios, 72 times and 20 periodicities), the single-cut algorithm performed 103.8 and 7.6 times faster on average than solving the full space equivalent deterministic problem and multi-cut algorithm, respectively. This suggests that the theoretical bounds for the maximum number of required iterations to reach complete convergence do not guarantee that the multi-cut L-shaped method will always have a faster computational solution times than the classical L-shaped method in this case.

The results support the conclusion that more complex formulations considering, for example, more items and more layers, for system (R, S), could be efficiently approached using the proposed methodology.

As for future research, we are currently investigating methods that blends single-cut and multi-cut versions for the case studied (less time solution with less interaction) and how to make improvements in the current model of Cunha et al. (2014) in order to be able to consider initial stock and partial backorder.

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