

## Cograph- $(k, \ell)$ Graph Sandwich Problem

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### ABSTRACT

A cograph is a graph without induced  $P_4$  (where  $P_4$  denotes an induced path with 4 vertices). A graph  $G$  is  $(k, \ell)$  if its vertex set can be partitioned into at most  $k$  independent sets and  $\ell$  cliques. Threshold graphs are cographs- $(1, 1)$ . Cographs- $(2, 1)$  are a generalization of threshold graphs and, as threshold graphs, they can be recognized in linear time. GRAPH SANDWICH PROBLEMS FOR PROPERTY  $\Pi$  ( $\Pi$ -SP) were defined by Golubic et al. as a natural generalization of RECOGNITION PROBLEMS. In this paper we show that, although COGRAPH-SP and THRESHOLD-SP are polynomially solvable problems, COGRAPH- $(2, 1)$ -SP and JOIN OF TWO THRESHOLDS-SP are NP-complete problems. As a corollary, we have that COGRAPH- $(1, 2)$ -SP is NP-complete as well. Using these results, we prove that COGRAPH- $(k, \ell)$  is NP-complete for  $k, \ell$  positive integers such that  $k + \ell \geq 3$ .

**KEYWORDS.** Graph Sandwich Problems, Cograph- $(k, \ell)$ , Cograph- $(2, 1)$ , Cograph- $(1, 2)$ , Join of Two Threshold Graphs.

**Main Area:** Theory and Algorithms on Graphs

### 1. Introduction

Introduced by Golubic, Kaplan and Shamir (GOLUBIC; KAPLAN; SHAMIR, 1995), GRAPH SANDWICH PROBLEMS arose as a natural generalization of RECOGNITION PROBLEMS. This decision problem can be formulated as follows:

GRAPH SANDWICH PROBLEM FOR PROPERTY  $\Pi$  ( $\Pi$ -SP)

*Input:* Two graphs  $G^1 = (V, E^1)$  and  $G^2 = (V, E^2)$  such that  $E^1 \subseteq E^2$ .

*Question:* Is there a graph  $G = (V, E)$  satisfying property  $\Pi$  and such that  $E^1 \subseteq E \subseteq E^2$  ?

Each edge in  $E^1$  is called *forced edge*, each edge of  $E^2 \setminus E^1$  is called *optional edge* and each edge in the complement graph of  $E^2$ , a forbidden edge. We will denote by  $G^3 = (V, E^3)$  the complement graph of  $G^2$ . We can then, rewrite GRAPH SANDWICH PROBLEMS as follows:

GRAPH SANDWICH PROBLEM FOR PROPERTY  $\Pi$  ( $\Pi$ -SP)

*Input:* A triple  $(V, E^1, E^3)$ , where  $E^1 \cap E^3 = \emptyset$ .

*Question:* Is there a graph  $G = (V, E)$  satisfying  $\Pi$  such that  $E^1 \subseteq E$  and  $E \cap E^3 = \emptyset$ ?

Given a property  $\Pi$ , we define its complementary property  $\bar{\Pi}$  as follows: for every graph  $G$ ,  $G$  satisfies  $\Pi$  if and only if  $\bar{G}$  satisfies  $\bar{\Pi}$  (GOLUMBIC; KAPLAN; SHAMIR, 1995). In this paper, we work with properties: “to be cograph-(2, 1)” and “to be cograph-(1, 2)”. Notice that the second is the complementary property of the former.

**Fact 1.1.** *There is a sandwich graph with property  $\Pi$  for the instance  $(V, E^1, E^3)$  if and only if there is a sandwich graph with property  $\bar{\Pi}$  for the instance  $(V, E^3, E^1)$ .*

*Perfect Graphs* attract a lot of attention in Graph Theory. In the seminal paper of GRAPH SANDWICH PROBLEMS (GOLUMBIC; KAPLAN; SHAMIR, 1995), Golumbic *et al.* worked only with subclasses of *perfect graphs*, for instance, *chordal graphs*, *cographs*, *threshold graphs* and *split graphs*, for which, except for chordal graphs, GRAPH SANDWICH PROBLEMS are polynomially solvable. They left some open problems, for example when  $\Pi$  is “to be strongly chordal” or “to be chordal bipartite”. Both were proved to be NP-complete by (FIGUEIREDO *et al.*, 2007) and (SRITHARAN, 2008) respectively. Moreover, many subclasses of perfect graphs were defined and studied in the context of GRAPH SANDWICH PROBLEMS. For example, CHORDAL- $(k, \ell)$ -SP and STRONGLY CHORDAL- $(k, \ell)$ -SP were proved NP-complete for  $k, \ell > 0$  such that  $k + \ell \geq 3$  and for  $k, \ell \geq 1$ , respectively (COUTO; FARIA; KLEIN, 2014).

In this work, we are particularly interested in one well known subclasse of perfect graphs: cographs.

**Definition 1.2.** (CORNEIL; LERCHS; Stewart Burlingham, 1981) *A cograph can be defined recursively as follows:*

1. *The trivial graph  $K_1$  is a cograph;*
2. *If  $G_1, G_2, \dots, G_p$  are cographs, then  $G_1 \cup G_2 \cup \dots \cup G_p$  is a cograph,*
3. *If  $G$  is a cograph, then  $\bar{G}$  is a cograph.*

There are some equivalent forms of characterizing a cograph (CORNEIL; LERCHS; Stewart Burlingham, 1981), but one of the best known is the characterization by forbidden subgraphs.

**Theorem 1.3.** (CORNEIL; LERCHS; Stewart Burlingham, 1981)

*A cograph is a  $P_4$ -free graph, i.e. a graph without induced paths with 4 vertices.*

Corneil in 1985 (CORNEIL; PERL; STEWART, 1985), presented the first, but not the only one, linear time algorithm to recognize cographs (BRETSCHER *et al.*, 2003; HABIB; PAUL, 2005).

*Threshold graphs* are a special case of cographs and split graphs. More formally, a graph is a threshold graph if and only if it is both a cograph and a split graph. This family was introduced and also characterized by Chvátal and Hammer in 1977 (CHVÁTAL; HAMMER, 1977).

**Theorem 1.4.** (CHVÁTAL; HAMMER, 1977) *For every graph  $G$ , the following three conditions are equivalent:*

1.  $G$  is threshold;
2.  $G$  has no induced subgraph isomorphic to  $2K_2, P_4$  or  $C_4$ ;
3. There is an ordering  $v_1, v_2, \dots, v_n$  of vertices of  $G$  and a partition of  $\{v_1, v_2, \dots, v_n\}$  into disjoint subsets  $P$  and  $Q$  such that:
  - Every  $v_j \in P$  is adjacent to all vertices  $v_i$  with  $i < j$ ,
  - Every  $v_j \in Q$  is adjacent to none of the vertices  $v_i$  with  $i < j$ .

Thus, threshold graphs can be constructed from a trivial graph  $K_1$  by repeated applications of the following two operations:

1. Addition of a single isolated vertex to the graph.
2. Addition of a single dominating vertex to the graph, i.e. a single vertex that is adjacent to each other vertex.

Brandstädt (BRANDSTÄDT, 1996) defined a special class of graphs named  $(k, \ell)$ -graphs, i.e. graphs whose vertex set can be partitioned into at most  $k$  independent sets and  $\ell$  cliques: a generalization of split graphs, which can be described as  $(1, 1)$ -graphs. Moreover, in (BRANDSTÄDT, 1996, 2005; BRANDSTÄDT; LE; SZYMCAK, 1998), it was proven that the recognition problem for this class of graphs is NP-complete for  $k$  or  $\ell$  at least 3 and polynomial, otherwise.  $(k, \ell)$  GRAPH SANDWICH PROBLEMS were fully classified with respect to the computational complexity, for integers  $k, \ell$ : the problem is NP-complete for  $k + \ell \geq 3$  and polynomial, otherwise (DANTAS; FIGUEIREDO; FARIA, 2004).

For cograph- $(k, \ell)$ , there is a characterization by forbidden subgraphs (BRAVO; KLEIN; NOGUEIRA, 2005; FEDER; HELL; HOCHSTÄTTLER, 2007; BRAVO *et al.*, 2011). Recently we provided a structural characterization and decomposition for cographs- $(2, 1)$  which leads us to a linear time algorithm to recognize this class of graphs, that can be considered, accordingly to the characterization, a generalization of threshold graphs (COUTO *et al.*, 2015). Before presenting this characterization, we make some helpful definitions.

**Definition 1.5.** The union of two graphs  $G = (V_G, E_G)$  and  $H = (V_H, E_H)$  is the union of their vertex and edge sets:  $G \cup H = (V_G \cup V_H, E_G \cup E_H)$ .

**Definition 1.6.** The disjoint union of two graphs  $G = (V_G, E_G)$  and  $H = (V_H, E_H)$  is the union of their vertex and edge sets when  $V_G$  and  $V_H$  are disjoint:  $G + H = (V_G + V_H, E_G + E_H)$ .

**Definition 1.7.** The join  $G \oplus H$  of two graphs  $G = (V_G, E_G)$  and  $H = (V_H, E_H)$  is their graph union with all edges that connect the vertices of  $G$  with the vertices of  $H$ , i.e.,  $G \oplus H = (V_G \cup V_H, E_G \cup E_H \cup \{uv : u \in V_G, v \in V_H\})$ .

**Theorem 1.8.** (COUTO *et al.*, 2015) Let  $G$  be a graph. Then the following are equivalent.

1.  $G$  is a cograph- $(2,1)$ .
2.  $G$  can be partitioned into a collection of maximal bicliques  $B = \{B_1, \dots, B_l\}$  and a clique  $K$  such that  $B_i = (X_i, Y_i)$  and  $V(K)$  is the union of non-intersecting sets  $K^1$  and  $K^2$  such that the following properties hold.
  - (a) There are no edges between vertices of  $B_i$  and  $B_j$  for  $i \neq j$ ;

- (b) Let  $L(v)$  be the list of bicliques in the neighborhood of  $v$ ,  $\forall v \in V$ .  
 $K^1 = \{v \in K | N(v) \cap B \subseteq B_1\} = K^{1,1} \cup K^{1,2}$  and  
 $K^2 = \{v \in K | L(v) \geq 2, B_i \in L(v) \Leftrightarrow B_i \subseteq N(v)\}$ , where  
 $K^{1,1} = \{v \in K^1 | vx \in E(G), \forall x \in X_1\}$  and  
 $K^{1,2} = K^1 \setminus K^{1,1}$  and it holds that  $uy \in E(G), \forall u \in K^{1,2}$  and  $y \in Y_1$ ;
- (c)  $G[X_1 \cup Y_1 \cup K^{1,1} \cup K^{1,2}]$  is the join of threshold graphs  $(K^{1,1}, Y_1)$  and  $(K^{1,2}, X_1)$ ;
- (d) There is an ordering  $v_1, v_2, \dots, v_{|K^2|}$  of  $K^2$ 's vertices such that  $N(v_i) \subseteq N(v_j), \forall i \leq j$  and  $N(v) \subseteq N(v_1), \forall v \in K^1$ .

3.  $G$  is either a join of two threshold graphs or it can be obtained from the join of two threshold graphs by the applications of any sequence of the following operations:

- Disjoint union with a biclique;
- Join with a single vertex.

After providing this characterization and considering that GRAPH SANDWICH PROBLEMS are not monotone with respect to their computational complexity, we got interested in COGRAPH-(2,1)-SP. Dealing with Theorem 1.8 to solve this problem, another interesting and motivating question arose: Given a property  $\Pi$  for which  $\Pi$ -SP is known to be polynomially solvable, is  $(\Pi \oplus \Pi)$ -SP also in P?

In this paper we answer this question negatively and we present the first NP-complete  $(\Pi \oplus \Pi)$ -SP, with  $\Pi$ -SP in P: JOIN OF TWO THRESHOLDS GRAPH SANDWICH PROBLEM. With this result in hands, we show, in section 3, that COGRAPH-(2,1)-SP and COGRAPH-(1,2)-SP are also NP-complete. In section 4, we prove that COGRAPH-( $k, \ell$ )-SP is NP-complete for  $k, \ell$  positive integers such that  $k + \ell \geq 3$ . These results corroborate the fact that GRAPH SANDWICH PROBLEMS are not monotone.

## 2. JOIN OF TWO THRESHOLDS GRAPH SANDWICH PROBLEM

In this section we prove that, JOIN OF TWO THRESHOLDS GRAPH SANDWICH PROBLEM is NP-complete, although THRESHOLD GRAPH SANDWICH PROBLEM is polynomially solvable (GOLUMBIC; KAPLAN; SHAMIR, 1995).

Proposition 2.1 was the key to prove that THRESHOLD-SP is a polynomial time solvable problem and it will be very helpful in this work.

**Proposition 2.1.** (GOLUMBIC; KAPLAN; SHAMIR, 1995) *Let  $(V, E^1, E^3)$  be a threshold sandwich instance and let  $v \in V$  be an isolated vertex in  $G^1$  or in  $G^3$ . There is a threshold sandwich for  $(V, E^1, E^3)$  if and only if there is a threshold sandwich for  $(V, E^1, E^3)_{V \setminus \{v\}}$ .*

Next we define JOIN OF TWO THRESHOLD GRAPH SANDWICH PROBLEM. We remark that one the thresholds of the join might be empty.

### JOIN OF TWO THRESHOLD GRAPH SANDWICH PROBLEM (JTT-SP)

*Input:* A triple  $(V, E^1, E^3)$ , where  $E^1 \cap E^3 = \emptyset$ .

*Question:* Is there a graph  $G = (V, E)$  which is a join of two threshold graphs such that  $E^1 \subseteq E$  and  $E \cap E^3 = \emptyset$ ?

To prove the main result of this section, stated below, we make a polynomial time reduction from the NP-complete problem MONOTONE NAE3SAT (GAREY; JOHNSON, 1979), which can be formulated as follows:

MONOTONE NOT ALL EQUAL 3-SATISFABILITY (MONOTONE NAE3SAT)

*Input:* A pair  $(X, C)$ , where  $X = \{x_1, \dots, x_n\}$  is the set of variables, and  $C = \{c_1, \dots, c_m\}$  is the collection of clauses over  $X$  such that each clause  $c \in C$  has exactly 3 positive literals.

*Question:* Is there a truth assignment for  $X$  such that each clause has at least one true and one false literals ?

**Theorem 2.2.** JOIN OF TWO THRESHOLDS GRAPH SANDWICH PROBLEM is an NP-complete problem.

*Proof.* In order to reduce MONOTONE NAE3SAT to JOIN OF TWO THRESHOLDS-SP we first construct a particular instance  $(V, E^1, E^3)$  of JTT-SP, from a generic instance  $(X, C)$  of MONOTONE NAE3SAT. Second, in Lemma 2.6 we prove that if there is a sandwich graph which is the join of two threshold graphs for  $(V, E^1, E^3)$ , then there is a truth assignment satisfying each clause of  $(X, C)$  such that in each clause we have at least one true and one false literals. Finally, in Lemma 2.8 we prove that if there is a truth assignment satisfying each clause of  $(X, C)$ , an instance of MONOTONE NAE3SAT, then there is a sandwich graph for  $(V, E^1, E^3)$  which is the join of two threshold graphs.  $\square$

**Remark 2.3.** Let  $G = (V, E)$  be a sandwich graph for  $(V, E^1, E^3)$  which is the join of two threshold graphs  $H_1, H_2$ . If  $e = xy \in E^3$ , then  $x, y$  are both either in  $H_1$  or in  $H_2$ .

**Remark 2.4.** If  $G$  is the join of two threshold graphs, then  $G$  is a cograph. Therefore,  $G$  has no induced  $P_4$ .

**Remark 2.5.** If  $G$  is the join of two threshold graphs  $H_1, H_2$  and  $G$  has an induced  $C_4 \{a, b, c, d, a\}$ , then  $\{a, b, c, d, a\}$  cannot be entirely contained in  $H_1$  or in  $H_2$ .

*Construction of the particular instance  $(V, E^1, E^3)$  for JTT-SP (see Figure 1):*

#### Variable gadget

- Vertices: For each variable  $x_i \in X$ ,  $i \in \{1, \dots, n\}$  add vertices  $x_1^i, y_1^i, x_2^i, y_2^i$ . For each time that a variable  $x_i \in X$ ,  $i \in \{1, \dots, n\}$  figures in a clause  $c_j$ ,  $j \in \{1, \dots, m\}$ , add vertices  $c_i^j, d_i^j, h_i^j$ .
- For  $i \in \{1, \dots, n\}$  and  $j \in \{1, \dots, m\}$ , add the following forced edges:  $\{x_1^i x_2^i, x_1^i y_2^i, x_2^i y_1^i, y_1^i y_2^i, c_i^j d_i^j\}$ .
- For  $i \in \{1, \dots, n\}$  and  $j \in \{1, \dots, m\}$ , add the following forbidden edges:  $\{x_1^i y_1^i, x_2^i y_2^i, y_1^i c_i^j, y_1^i d_i^j, y_1^i h_i^j, c_i^j h_i^j, d_i^j h_i^j\}$

#### Clause gadget

- Vertices: For each clause  $c_j \in C$ ,  $j \in \{1, \dots, m\}$ , add vertices  $r_1^j, r_2^j, r_3^j$ .
- For  $j \in \{1, \dots, m\}$  add the following forced edges:  $\{r_1^j r_2^j, r_1^j r_3^j, r_2^j, r_3^j\}$ .  
For each clause  $c_j = (l_1^j \vee l_2^j \vee l_3^j) \in C$ , add to  $E^1$  edges  $h_1^j r_1^j, h_2^j r_2^j, h_3^j r_3^j$ .
- Forbidden edges:  
For each clause  $c_j = (l_1^j \vee l_2^j \vee l_3^j) \in C$ , add to  $E^3$  edges  $h_1^j r_2^j, h_2^j r_3^j, h_3^j r_1^j$ .

**Lemma 2.6.** If there is a sandwich graph  $G = (V, E)$  which is the join of two threshold graphs for the particular instance  $(V, E^1, E^3)$  constructed above, then there is a truth assignment satisfying each clause of  $(X, C)$ , a generic instance of MONOTONE NAE3SAT.

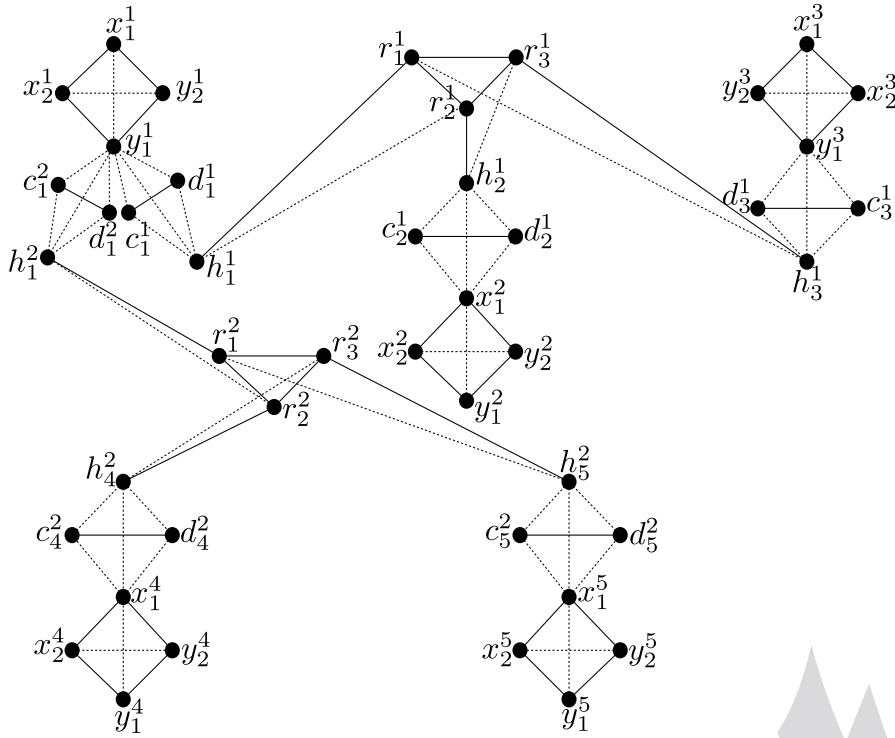


Figure 1: Example of a particular instance  $(V, E^1, E^3)$  of JTT-SP obtained from the instance of MONOTONE NAE3SAT:  $I = (X, C) = (\{x_1, x_2, x_3, x_4, x_5\}, (x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee x_4 \vee x_5))$ . Solid edges are forced  $E^1$ -edges, dashed edges are forbidden  $E^3$ -edges and omitted edges are optional edges.

*Proof.* Suppose there exists a sandwich graph  $G = (V, E)$  which is the join of two threshold graphs  $H_1, H_2$ . We define the truth assignment for  $(X, C)$ : for  $i \in \{1, \dots, n\}$ , variable  $x_i$  is true if and only if  $x_1^i, y_1^i \in H_1$ . Suppose that for some  $j \in \{1, \dots, m\}$ , each literal of clause  $c_j = (l_1^j \vee l_2^j \vee l_3^j)$  is false. If  $l_q^j = x_i$ ,  $q \in \{1, 2, 3\}$  then, since  $c_j$  is false,  $x_1^i, y_1^i \notin H_1$ . Thus,  $x_1^i, y_1^i \in H_2$  and, by Remarks 2.3 and 2.5,  $x_2^i, y_2^i \in H_1$ . By Remark 2.3, we can also affirm that  $c_i^j, d_i^j, h_i^j, r_1^j, r_2^j, r_3^j$  belong to  $H_2$ .

**Claim 2.7.** *If three vertices  $x, y, z$  belong to the same threshold graph such that  $xy \in E^1$  and  $zx, zy \in E^3$ , then  $z \in S$ , where  $S$  is the independent set of the threshold graph.*

*Proof of Claim 2.7:* Since  $x, y, z$  are all in the same threshold and  $xy \in E^1$ , then either  $x \in K$  or  $y \in K$ , where  $K$  is the clique of the threshold graph. Thus,  $z$  cannot be in  $K$  and, consequently,  $z \in S$ .  $\square$

Therefore, by Claim 2.7, we have that  $h_i^j \in S_2$ , where  $S_2$  is the independent set of  $H_2$ . Note that, in this case,  $h_1^j r_1^j r_2^j h_2^j$  would be a  $P_4$  in  $G$ , what is a contradiction by Remark 2.4. So,  $h_2^j r_1^j \in E$ . But in this case,  $h_2^j r_1^j r_3^j h_3^j$  is an induced  $P_4$  in  $G$  which we cannot destroy, a contradiction.

The case where all literals are true is also impossible, by symmetry.  $\square$

**Lemma 2.8.** *If there is a truth assignment satisfying each clause of  $(X, C)$ , a generic instance of MONOTONE NAE3SAT, then there is a sandwich graph  $G = (V, E)$  which is a join of two threshold graphs for  $(V, E^1, E^3)$ .*

*Proof.* Suppose there is a truth assignment that satisfies  $(X, C)$ . We define a join of two thresholds graph that is the sandwich graph for the particular instance  $(V, E^1, E^3)$  of JTT-SP associated with

MONOTONE NAE3SAT instance  $(X, C)$ .

We have two types of clauses in  $C$ : those with two true and one false literals (*type1 clause*) and those with one true and two false literals (*type2 clause*).

If variable  $x_i$  is true (resp. false), then include  $x_1^i, y_1^i$  in  $H_1$  (resp.  $H_2$ ) and  $x_2^i, y_2^i$  in  $H_2$  (resp.  $H_1$ ), for  $i \in \{1, \dots, n\}$ . By Remarks 2.3 and 2.5 and Claim 1, we have that, for each clause  $c_j$ ,  $j \in \{1, \dots, m\}$  in which  $x_i$  figures,  $c_i^j, d_i^j, h_i^j \in H_1$  (resp.  $H_2$ ).

If we have a type1 clause (resp. type2 clause)  $c_j = (l_1^j \vee l_2^j \vee l_3^j)$  and supposing, without loss of generality, that  $l_1^j$  and  $l_2^j$  are true literals (resp. false literals), then  $r_2^j$  and  $r_3^j$  are in  $H_1$  (resp.  $H_2$ ) and  $r_1^j$  is in  $H_2$  (resp.  $H_1$ ). With  $H_1$  and  $H_2$  well-defined by the truth assignment for  $(X, C)$ , it remains to prove that  $(V_{H_1}, E_{H_1}^1, E_{H_1}^3)$  and  $(V_{H_2}, E_{H_2}^1, E_{H_2}^3)$  are YES instances for THRESHOLD-SP. In order to show it, we state the following two Claims.

**Claim 2.9.** *The local analysis of  $H_1$  and  $H_2$ , i.e., to analyze if  $(V_{H_1}, E_{H_1}^1, E_{H_1}^3)$  and  $(V_{H_2}, E_{H_2}^1, E_{H_2}^3)$  are YES instances for THRESHOLD-SP considering a subgraph induced by vertices related to only one clause, for each clause, is equivalent to the global analysis, i. e., to analyze if  $(V_{H_1}, E_{H_1}^1, E_{H_1}^3)$  and  $(V_{H_2}, E_{H_2}^1, E_{H_2}^3)$  are YES instances for THRESHOLD-SP considering all of their vertices.*

**Proof of Claim 2.9.** First consider vertices  $x_1^i, y_1^i, x_2^i, y_2^i$ . By construction of  $(V, E^1, E^3)$ , both pairs of vertices cannot be together in  $H_1$  or in  $H_2$ . Then, it is clear that: first, when we look locally to the induced subgraphs they belong to, they are isolated in there; second, they can be related to a lot of clauses but they are still isolated when look globally to  $G^1[V_{H_1}]$  or  $G^1[V_{H_2}]$ . Thus, if we remove them based in a local analysis, we are sure that this removal is allowed considering  $H_1$  or  $H_2$  entirely.

Each remaining vertex of  $V$  is only related to one clause. Moreover, between variable gadgets we do not have forbidden edges. We can say the same about clause gadgets. Therefore, if one of these vertices is considered isolated in a local analysis in  $G^1[V_{H_i}]$  or in  $G^3[V_{H_i}]$ ,  $i = 1, 2$ , then it is also isolated when we consider the entire graph  $G^1[V_{H_i}]$  or  $G^3[V_{H_i}]$ ,  $i = 1, 2$ .  $\square$

**Claim 2.10.**  $(V_{H_1}, E_{H_1}^1, E_{H_1}^3)$  and  $(V_{H_2}, E_{H_2}^1, E_{H_2}^3)$  are YES instances for THRESHOLD-SP.

**Proof of Claim 2.10.** We will first work with  $(V_{H_1}, E_{H_1}^1, E_{H_1}^3)$ . Accordingly to Proposition 2.1 and Claim 2, we must describe an isolated vertex elimination ordering for  $V_{H_1}$  considering  $H_1$  induced by vertices related to one clause. Since we have two kinds of clauses, we will deal with them separately. In order to simplify the following analysis and, since we are making a reduction from MONOTONE NAE3SAT, will we assume that all variables are literals.

- Type1 clause:  $c_j = (x_1 \vee x_2 \vee x_3)$ , where  $x_1$  and  $x_2$  are true.

In this case,  $\{x_1^i, y_1^i, x_2^3, y_2^3, c_i^j, d_i^j, h_i^j, r_2^j, r_3^j\} \in V_{H_1}$ , for  $i = 1, 2$ . Consider the sets below:

- $I_1 = \{x_1^i, y_1^i, x_2^3, y_2^3\}$ , for  $i = 1, 2$ ;
- $I_2 = \{h_1^j\}$ ;
- $I_3 = \{c_1^j, d_1^j\}$ ;
- $I_4 = \{r_2^j\}$ ;
- $I_5 = \{h_2^j, r_3^j\}$ ;
- $I_6 = \{c_2^j, d_2^j\}$

We affirm that, following the set order described and any order inside sets above, we define an isolated vertex elimination ordering.

First,  $I_1$  is an independent set in  $G^1[\bigcup_{q=1,\dots,6} I_q]$ . Moreover, the only neighbors of  $x_1^i$  and  $y_1^i$  in  $G^1[V_{H_1}]$  are  $x_2^i, y_2^i$  (and vice-versa), which do not belong to  $V_{H_1}$ , for  $i = 1, 2$ . So,  $x_1^i, y_1^i, x_2^i, y_2^i$  for  $i = 1, 2$  are isolated vertices in  $G^1[\bigcup_{q=1,\dots,6} I_q]$  and then they can be removed. Let  $A$  be the resulting graph. The only neighbor of  $h_1^j$  is  $r_1^j$  which is not in  $A$ . So, we can also remove  $h_1^j$  of  $A$ , obtaining  $B$ . In  $I_3$  we have  $c_1^j, d_1^j$ , which are not isolated in  $B$ , since they are adjacent. But, in  $G^3[V_B \cap \bigcup_{q=3,\dots,6} I_q]$ , they are isolated. So, they can be deleted and we call  $C$  the resulting graph. Following the proposed ordering, we are able to remove  $r_2^j$ , which is clearly isolated in  $G^3[V_C \cap \bigcup_{q=4,\dots,6} I_q]$ . Let  $D$  be the graph we obtain after this removal.  $D$  has two isolated vertices  $h_2^j$  and  $r_3^j$  that can be removed, generating graph  $F$ . Finally,  $F$  is a threshold graph and we finish the proof.

- Type2 clause:  $c_j = (x_1 \vee x_2 \vee x_3)$ , where  $x_1$  and  $x_2$  are false.

In this case,  $\{x_1^3, y_1^3, x_2^i, y_2^i, c_3^j, d_3^j, h_3^j, r_1^j\} \in V_{H_1}$ , for  $i = 1, 2$ . Consider the sets below:

- $I_1 = \{x_1^3, y_1^3, x_2^i, y_2^i, h_3^j, r_1^j\}$ , for  $i = 1, 2$ ;
- $I_2 = \{c_3^j, d_3^j\}$ ;

We affirm that, following the set order described and any order inside sets above, we define an isolated vertex elimination ordering.

Note that  $I_1$  is an independent set in  $G^1[I_1 \cup I_2]$  and each vertex in  $I_1$  is isolated in  $G^1[I_1 \cup I_2]$ . Thus, we can remove all of them from  $V_{H_1}$  obtaining graph  $A$  that is a threshold graph.

The local analysis of  $(V_{H_2}, E_{H_2}^1, E_{H_2}^3)$  follows similarly. Observe that, when we analyze a type1 clause, we have the same isolated vertex elimination ordering we presented above for the type2 clause's case, by symmetry. When we analyze a type2 clause, we have the same isolated vertex elimination ordering we presented above in the case of a type1 clause, again by symmetry.

Thus, we proved that  $(V_{H_1}, E_{H_1}^1, E_{H_1}^3)$  and  $(V_{H_2}, E_{H_2}^1, E_{H_2}^3)$  are YES instances for THRESHOLD-SP.  $\square$

To finish the proof of Lemma 2.8, after obtaining sandwich graphs for instances  $(V_{H_1}, E_{H_1}^1, E_{H_1}^3)$  and  $(V_{H_2}, E_{H_2}^1, E_{H_2}^3)$ , it remains to add all edges between vertices of  $H_1$  and  $H_2$  in order to obtain a join. We remark that all of these edges are allowed, since by Remark 2.3, forbidden edges assign vertices to the same threshold.  $\square$

### 3. COGRAPH-(2, 1) GRAPH SANDWICH PROBLEM

In this section we prove that COGRAPH-(2, 1)-SP is NP-complete as an application of cograph-(2, 1) structural characterization we presented in (COUTO *et al.*, 2015). This problem can be formulated as follows:

COGRAPH-(2, 1) GRAPH SANDWICH PROBLEM (COGRAPH-(2, 1)-SP)

*Input:* A triple  $(V, E^1, E^3)$ , where  $E^1 \cap E^3 = \emptyset$ .

*Question:* Is there a graph  $G = (V, E)$  which is a cograph-(2, 1) and such that  $E^1 \subseteq E$  and  $E \cap E^3 = \emptyset$ ?

**Theorem 3.1.** COGRAPH-(2, 1) GRAPH SANDWICH PROBLEM is NP-complete.



*Proof.* Clearly, COGRAPH-(2, 1)-SP is a problem in NP. In order to prove that COGRAPH-(2, 1)-SP is an NP-complete problem, we will make a polynomial time reduction from the NP-complete problem JTT-SP (proved in Theorem 2.2). Let  $(V, E^1, E^3)$  be a generic instance for JTT-SP. We assume that there is no vertex  $u$  such that  $N_{G^3}(u) = \emptyset$ , since the removal of this kind of vertex does not affect the property of being a PP-JTT. First, we will construct a particular instance  $(V', E'^1, E'^3)$  for COGRAPH-(2, 1)-SP. Second, we prove that if there is a join of two thresholds sandwich graph for  $(V, E^1, E^3)$ , then there is a cograph-(2, 1) sandwich graph for  $(V', E'^1, E'^3)$ . Third, we prove that if there is a cograph-(2, 1) sandwich graph for  $(V', E'^1, E'^3)$ , then there is a join of two thresholds sandwich graph for  $(V, E^1, E^3)$ .  $\square$

*Construction of the particular instance  $(V', E'^1, E'^3)$  for COGRAPH-(2, 1)-SP:*

- $V' = V \cup \{a, b, c, d\}$ ,
- $E'^1 = E^1 \cup \{ab, bc, cd, da\} \cup \{xy | x \in V, y \in \{a, b\}\}$
- $E'^3 = E^3 \cup \{ac, bd\}$

The next Lemmas complete Theorem 3.1's proof.

**Lemma 3.2.** *Let  $G = (V, E)$  be a graph with an universal vertex  $u$ .  $G$  is a join of two thresholds graph if and only if  $G \setminus \{u\}$  is a join of two thresholds graph.*

*Proof.* Suppose  $G$  is a join of two thresholds graph. Since being a join of two thresholds graph is an hereditary property for induced subgraphs,  $G \setminus \{u\}$  is also a join of two thresholds graph. Conversely, suppose  $G \setminus \{u\}$  is a join of two thresholds graph. If we add an universal vertex  $u'$  to  $G \setminus \{u\}$ , then  $u$  can be assigned to any threshold graph of  $G \setminus \{u\}$  without ruin its property of being a threshold graph (CHVÁTAL; HAMMER, 1977). The resulting graph is still a join of two thresholds graph, since independently to what threshold  $u'$  was assigned, he will be adjacent to each vertex of the other threshold.  $\square$

**Lemma 3.3.** *If there is a join of two thresholds sandwich graph for  $(V, E^1, E^3)$ , then there is a cograph-(2, 1) sandwich graph for  $(V', E'^1, E'^3)$ .*

*Proof.* Indeed, if there is a join of two thresholds sandwich graph  $G = (V, E)$  for  $(V, E^1, E^3)$  such that  $H_1$  and  $H_2$  are the two threshold graphs of  $G$ , by Lemma 3.2, we may assume that  $G$  has no universal vertex. We define  $G' = (V', E')$  where  $V' = V \cup \{a, b, c, d\}$ ,  $E' = E \cup \{ab, bc, cd, da\} \cup \{xy | x \in V, y \in \{a, b\}\} \cup E^* \cup E^{**}$ , where  $E^* = \{cy | \forall y \in H_2\}$  and  $E^{**} = \{dz | \forall z \in H_1\}$ . Clearly,  $G'$  is a sandwich graph for  $(V', E'^1, E'^3)$ . Therefore, to finish the proof, we affirm that  $G'$  is a cograph-(2, 1) and we prove it by showing that  $G'$  is still a join of two thresholds graph. We assign  $a, c$  to  $H_1$ , since they must be together in the same threshold by Lemmas 2.3 and 2.5. By the same reason, we assign  $b, d$  to  $H_2$ .  $a$  and  $b$  (resp.  $c$  and  $d$ ) are universal (resp. isolated) vertices inside the threshold they are added to. So, their additions do not ruin graphs property of being thresholds (CHVÁTAL; HAMMER, 1977). Furthermore,  $a, c$  are adjacent to every vertex of  $H_2$  and  $b, d$  are adjacent to every vertex of  $H_1$ . Thus  $G'$  is a join of two thresholds graph and, hence, a cograph-(2, 1).  $\square$

**Lemma 3.4.** *If there is a cograph-(2, 1) sandwich for  $(V', E'^1, E'^3)$ , then there is a join of two thresholds sandwich graph for  $(V, E^1, E^3)$ .*

*Proof.* Let  $G' = (V', E')$  be a cograph-(2, 1) sandwich for  $(V', E'^1, E'^3)$ . By Theorem 1.8,  $G'$  has either an isolated biclique, or an universal vertex or  $G$  is a join of two thresholds graph. Since  $G'^1$  is connected,  $G'$  cannot have an isolated biclique. Moreover, by construction,  $G'^3$  has no

isolated vertex, so  $G'$  does not have universal vertices. Hence,  $G'$  is a join of two thresholds graph. Consider the graph  $G = (V^*, E^*)$  constructed as follows:  $V^* = V' \setminus \{a, b, c, d\}$  and  $E^* = E' \setminus (\{ab, bc, cd, da\} \cup \{xy | x \in V, y \in \{a, b\}\} \cup E^{**} \cup E^{***})$ , where  $E^{**} = \{cy | \forall y \in H_2\}$  and  $E^{***} = \{dz | \forall z \in H_1\}$ . Since being a join of two thresholds graph is an hereditary property for induced subgraphs,  $G$  is a join of two thresholds graph. □

Since the complement of a cograph-(2, 1) is a cograph-(1, 2) and accordingly to Fact 1.1, we can state Corollary 3.5.

**Corollary 3.5.** COGRAPH-(1, 2) GRAPH SANDWICH PROBLEM is NP-complete.

#### 4. COGRAPH-( $k, \ell$ ) GRAPH SANDWICH PROBLEM

The goal of this section is to generalize the results obtained in the previous one. We prove that COGRAPH-( $k, \ell$ )-SP, for  $k$  and  $\ell$  positive integers such that  $k + \ell \geq 3$ , which is formulated below, is an NP-complete problem.

COGRAPH-( $k, \ell$ ) GRAPH SANDWICH PROBLEM (COGRAPH-( $k, \ell$ )-SP)

*Input:* A triple  $(V, E^1, E^3)$ , where  $E^1 \cap E^3 = \emptyset$ .

*Question:* Is there a graph  $G = (V, E)$  which is a cograph-( $k, \ell$ ) and such that  $E^1 \subseteq E$  and  $E \cap E^3 = \emptyset$ ?

Next, we state the main Theorem of this section. We prove it by induction on  $k$  and  $\ell$ .

**Theorem 4.1.** For  $k, \ell$  positive integers such that  $k + \ell \geq 3$ , COGRAPH-( $k, \ell$ )-SP is NP-complete.

*Proof.* COGRAPH-( $k, \ell$ )-SP is in NP, since we can recognize a cograph-( $k, \ell$ ) in linear time (BRAVO; KLEIN; NOGUEIRA, 2005) and we can decide in polynomial time if a graph is a sandwich graph for  $(V, E^1, E^3)$ . We prove that COGRAPH-( $k, \ell$ ) is NP-complete, for  $k, \ell$  positive integers such that  $k + \ell \geq 3$ , by induction on  $k$  and  $\ell$  using Theorem 3.1 and Corollary 3.5 as induction bases and Lemmas 4.2 and 4.5 as inductive steps. □

**Lemma 4.2.** If COGRAPH-( $k, \ell$ )-SP is NP-complete, then COGRAPH-( $k + 1, \ell$ )-SP is NP-complete.

*Proof.* Let  $(V, E^1, E^3)$  be a generic instance of COGRAPH-( $k, \ell$ )-SP. We construct a particular instance  $(V', E^{1'}, E^{3'})$  for COGRAPH-( $k + 1, \ell$ )-SP as follows:

- $V' = V \cup \{a_1, \dots, a_{\ell+1}\};$
- $E^{1'} = E^1 \cup \{a_i v\}, i = 1, \dots, \ell + 1$  and  $\forall v \in V;$
- $E^{3'} = E^3 \cup \{a_i a_j\}, i \neq j, i, j \in \{1, \dots, \ell + 1\}.$

**Claim 4.3.** If there is a cograph-( $k, \ell$ ) sandwich graph for  $(V, E^1, E^3)$ , then there is a cograph-( $k + 1, \ell$ ) sandwich graph for  $(V', E^{1'}, E^{3'})$ .

**Proof of Claim 4.3.** Let  $G = (V, E)$  be a cograph-( $k, \ell$ ) sandwich graph for  $(V, E^1, E^3)$ . We affirm that  $G' = (V', E')$  is a cograph-( $k + 1, \ell$ ) sandwich graph for  $(V', E^{1'}, E^{3'})$ , where  $E' = E \cup E^{1'}$ . Indeed, it is clear that  $G'$  is a sandwich graph for  $(V', E^{1'}, E^{3'})$ . Moreover,  $G'$  is a cograph, since it was obtained by a join of cographs, and we can assign every added vertex to a new independent set in  $G'$ . □

**Claim 4.4.** If there is a cograph-( $k + 1, \ell$ ) sandwich graph for  $(V', E^{1'}, E^{3'})$ , then there is a cograph-( $k, \ell$ ) sandwich graph for  $(V, E^1, E^3)$ .

**Proof of Claim 4.4.** Let  $G'$  be a cograph- $(k + 1, \ell)$  sandwich graph for  $(V', E^{1'}, E^{3'})$ . We affirm that  $G = (V, E)$  is a cograph- $(k, \ell)$  sandwich graph for  $(V, E^1, E^3)$ , where  $E = E' \setminus \{a_1, \dots, a_{\ell+1}\}$ .  $G$  is, clearly, a sandwich graph for  $(V, E^1, E^3)$ . Notice that  $G$  is a cograph, since being a cograph is an hereditary property for induced subgraphs. Moreover, observe that, in  $G'$ , at least one vertex of  $\{a_1, \dots, a_{\ell+1}\}$  was in an independent set, which, by construction, could not contain a vertex of  $V$ . Thus, by removing the set  $\{a_1, \dots, a_{\ell+1}\}$ , we obtain a  $(k, \ell)$ -graph.  $\square$

This finishes Lemma 4.2's proof.  $\square$

**Lemma 4.5.** *If COGRAPH- $(k, \ell)$ -SP is NP-complete, then COGRAPH- $(k, \ell + 1)$ -SP is NP-complete.*

*Proof.* Let  $(V, E^1, E^3)$  be a generic instance of COGRAPH- $(k, \ell)$ -SP. We construct a particular instance  $(V', E^{1'}, E^{3'})$  for COGRAPH- $(k, \ell + 1)$ -SP as follows:

- $V' = V \cup \{a_1, \dots, a_{k+1}\}$ ;
- $E^{1'} = E^1 \cup \{a_i a_j\}, i \neq j, i, j \in \{1, \dots, k + 1\}$ ;
- $E^{3'} = E^3 \cup \{a_i v\}, i = 1, \dots, k + 1$  and  $\forall v \in V$ .

**Claim 4.6.** *If there is a cograph- $(k, \ell)$  sandwich graph for  $(V, E^1, E^3)$ , then there is a cograph- $(k, \ell + 1)$  sandwich graph for  $(V', E^{1'}, E^{3'})$ .*

**Proof of Claim 4.6.** Let  $G = (V, E)$  be a cograph- $(k, \ell)$  sandwich graph for  $(V, E^1, E^3)$ . We affirm that  $G' = (V', E')$  is a cograph- $(k, \ell + 1)$  sandwich graph for  $(V', E^{1'}, E^{3'})$ , where  $E' = E \cup E^{1'}$ . Indeed, it is clear that  $G'$  is a sandwich graph for  $(V', E^{1'}, E^{3'})$ . Moreover,  $G'$  is a cograph, since it was obtained by the union of cographs, and we can assign every added vertex to a new clique in  $G'$ .  $\square$

**Claim 4.7.** *If there is a cograph- $(k, \ell + 1)$  sandwich graph for  $(V', E^{1'}, E^{3'})$ , then there is a cograph- $(k, \ell)$  sandwich graph for  $(V, E^1, E^3)$ .*

**Proof of Claim 4.7.** Let  $G'$  be a cograph- $(k, \ell + 1)$  sandwich graph for  $(V', E^{1'}, E^{3'})$ . We affirm that  $G = (V, E)$  is a cograph- $(k, \ell)$  sandwich graph for  $(V, E^1, E^3)$ , where  $E = E' \setminus \{a_1, \dots, a_{k+1}\}$ .  $G$  is, clearly, a sandwich graph for  $(V, E^1, E^3)$ . Notice that  $G$  is a cograph, since being a cograph is an hereditary property for induced subgraphs. Moreover, observe that, in  $G'$ , at least one vertex of  $\{a_1, \dots, a_{k+1}\}$  was in a clique, which, by construction, could not contain a vertex of  $V$ . Thus, by removing the set  $\{a_1, \dots, a_{k+1}\}$ , we obtain a  $(k, \ell)$ -graph.  $\square$

This finishes Lemma 4.5's proof.  $\square$

## 5. Conclusions

In this paper, we analyzed the computational complexity of COGRAPH- $(2, 1)$ -SP and COGRAPH- $(1, 2)$ -SP as applications of the structural characterization and decomposition for cographs- $(2, 1)$  and  $(1, 2)$  (COUTO *et al.*, 2015). We showed that, although THRESHOLD-SP and COGRAPH-SP are polynomially solvable problems (GOLUMBIC; KAPLAN; SHAMIR, 1995), JOIN OF TWO THRESHOLDS-SP and consequently COGRAPH- $(2, 1)$ -SP and COGRAPH- $(1, 2)$ -SP are NP-complete ones, contradicting all natural feelings around two well known classes of graphs. Moreover, we generalized these results, proving that COGRAPH- $(k, \ell)$ -SP is NP-complete for  $k$  and  $\ell$  positive integers such that  $k + \ell \geq 3$ .

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