The Carathéodory number of the P_3 convexity of Cartesian product of graphs

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ABSTRACT

From Carathéodory's theorem arises the definition of the Carathéodory number for graphs. This number is well-known for monophonic and triangle-path convexities and it has been studied in P_3 and geodetic convexities. However, in the last two there are not many results for Cartesian product. In this paper we determine the Carathéodory number in P_3 convexity of the following Cartesian products: $K_n \square K_m$, $P_n \square K_m$, and $K_{1,n} \square K_m$, where K_m is the complete graph with m vertices, $K_{1,n}$ is the star with n leaves and P_n is a path with n vertices. Also, we present a recursive way to construct a Carathéodory set in $T_h \square K_m$, where T_h is a full binary tree with height h.

KEYWORDS. Carathéodory number. P_3 Convexity. Cartesian Product.

Main area: Theory and Algorithms in Graphs (TAG)

1. Introduction

Graph convexities are a well studied topic. For a finite, simple, and undirected graph G with vertex set $V(G)$, a *graph convexity* on $V(G)$ is a collection $\mathcal{C}$ of subsets of $V(G)$ such that

- $\emptyset, V(G) \in \mathcal{C}$ and
- $\mathcal{C}$ is closed under intersections.

The sets in $\mathcal{C}$ are called *convex sets* and the *convex hull* in $\mathcal{C}$ of a set $S \subseteq V(G)$ is the smallest set $H_{\mathcal{C}}(S)$ in $\mathcal{C}$ containing S.

Several well known graph convexities $\mathcal{C}$ are defined using some set $\mathcal{P}$ of paths of the underlying graph G. In this case, a subset S of $V(G)$ is convex, that is, belongs to $\mathcal{C}$, if for every path P in $\mathcal{P}$ whose end vertices belong to S also every vertex of P belongs to S. When $\mathcal{P}$ is the set of all shortest paths in G, this leads to the *geodetic convexity* [Cáceres et al., 2006; Dourado et al., 2010a; Everett and Seidman, 1985; Farber and Jamison, 1987]. The *monophonic convexity* is defined by considering as $\mathcal{P}$ the set of all induced paths of G [Dourado et al., 2010b; Duchet, 1988]. Similarly, if $\mathcal{P}$ is the set of all triangle paths in G, then $\mathcal{C}$ is the *triangle path convexity* [Changat and Mathew, 1999]. Here we consider the P_3 *convexity* of G, which is defined when $\mathcal{P}$ is the set of all paths of length two. The P_3 convexity was first considered for directed graphs [Erdös et al., 1972; Moon, 1972; Parker et al., 2008; Varlet, 1976]. For undirected graphs, the P_3 convexity was studied in [Barbosa et al., 2012; Centeno et al., 2011; Coelho et al., 2014; Duarte et al., 2017].

A famous result about convex sets in $\mathbb{R}^d$ is *Carathéodory's theorem* [Carathéodory, 1911]. It states that every point u in the convex hull of a set $S \subseteq \mathbb{R}^d$ lies in the convex hull of a subset F of S of order at most $d+1$. Let G be a graph and let $\mathcal{C}$ be a graph convexity on $V(G)$. The *Carathéodory number* of $\mathcal{C}$ is the smallest integer c such that for every set S of $V(G)$ and every vertex u in $H_{\mathcal{C}}(S)$, there is a set $F \subseteq S$ with $|F| \leq c$ and $u \in H_{\mathcal{C}}(F)$. A set $S \subseteq V(G)$ is a *Carathéodory set* of $\mathcal{C}$ if the set $\partial H_{\mathcal{C}}(S)$ defined as $H_{\mathcal{C}}(S) \setminus \bigcup_{u \in S} H_{\mathcal{C}}(S \setminus \{u\})$ is not empty. This notion allows an alternative definition of the *Carathéodory number* of $\mathcal{C}$ as the largest cardinality of a Carathéodory set of $\mathcal{C}$. Considering $\mathcal{C}$ the P_3 convexity, in Figure 1 we have a graph G and $S \subseteq V(G)$ with $S = \{a, d, f\}$. In this case $H_{\mathcal{C}}(S) = V(G) \setminus \{h\}$ and $g \in \partial H_{\mathcal{C}}(S)$. Then S is a Carathéodory set of G.

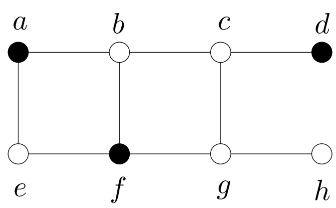


Figure 1: Graph G with a Carathéodory set of cardinality three.

The Carathéodory number was determined for several graph convexities. The Carathéodory number of the monophonic convexity of a graph G is 1 if G is complete and 2 otherwise [Duchet, 1988]. The Carathéodory number of the triangle path convexity of G is 2 whenever G has at least one edge [Changat and Mathew, 1999]. It is known that the maximum Carathéodory number of the P_3 convexity of a multipartite tournament is 3 [Parker et al., 2008]. Some general results concerning the Carathéodory number of the P_3 convexity are shown in [Barbosa et al., 2012]. On the one hand, [Barbosa et al., 2012] contains efficient algorithms to determine the Carathéodory number of the P_3 convexity of trees and, more generally, block graphs. On the other hand, it is NP-hard to determine the Carathéodory number of the P_3 convexity of bipartite graphs [Barbosa et al., 2012]. In [Dourado et al., 2013] it was determined that it is NP-hard to determine the Carathéodory number in the geodetic convexity. Lastly, [Duarte et al., 2017] showed that is NP-hard to determine the Carathéodory number of the P_3 convexity of complementary prisms and determined the Carathéodory number of complementary prims of trees.

Since a graph G uniquely determines its P_3 convexity $\mathcal{C}$, we speak of a Carathéodory set of G and the Carathéodory number $c(G)$ of G. Furthermore, we write $H_G(S)$ and $\partial H_G(S)$ instead of $H_{\mathcal{C}}(S)$ and $\partial H_{\mathcal{C}}(S)$, respectively.

In the present paper we exclusively study the Carathéodory number of P_3 convexity of some Cartesian product of graphs. We determine the Carathéodory number of the following Cartesian products: $K_n \square K_m$, $P_n \square K_m$, and $K_{1,n} \square K_m$, where K_m is the complete graph with m vertices, $K_{1,n}$ is the star with n leaves and P_n is a path with n vertices. Also, we present a recursive way to construct a Carathéodory set in $T_h \square K_m$, where T_h is a full binary tree with height h.

For a vertex u of G, its neighbourhood is denoted $N_G(u)$ and its *closed neighbourhood* denoted $N_G[u]$ is the set $N_G(u) \cup \{u\}$. For a set U of vertices of G, let $N_G(U) = \bigcup_{u \in U} N_G(u) \setminus U$ and $N_G[U] = N_G(U) \cup U$. The set $\{1, 2, \ldots, n\}$ is denoted by $[n]$.

2. Preliminaries on Cartesian product

The Cartesian product of two graphs G and H, denoted by $G \square H$, is the graph with vertex set $V(G \square H) = V(G) \times V(H)$ and edge set $E(G \square H)$ satisfying the following condition: $(u, u')(v, v') \in E(G \square H)$ if and only if

- either $u = v$ and $u'v' \in E(H)$ or
- $u' = v'$ and $uv \in E(G)$.

Figure 2 shows $C_3 \square C_5$, the Cartesian product of cycles C_3 and C_5.

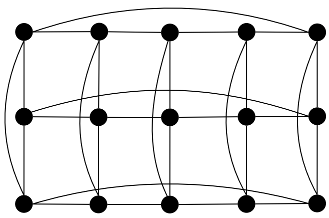


Figure 2: The Cartesian product $C_3 \square C_5$.

For convenience, we refer to the subgraph of $G \square H$ induced by $V(G) \square \{y\}$ (or $V(H) \square \{x\}$) as the G-layer (or H-layer) through y (or x). We denote $V(G) \square \{y\}$ (or $V(H) \square \{x\}$) by G^y (or H^x). The projection of S onto G is the set of vertices $a \in V(G)$ for which there exists a vertex $(a, v) \in S$. In Figure 3 we have the projection of a subset of vertices of $V(G \square H)$ onto G. Similarly, the projection of S onto H is the set of vertices $v \in V(H)$ for which there exists a vertex $(a, v) \in S$.

3. Results

We start by stating a result of [Barbosa et al., 2012] that collects several useful elementary properties of Carathéodory sets.

Proposition 1. *[Barbosa et al., 2012] Let G be a graph and let S be a Carathéodory set of G.*

a) G has order at least 2 and is either complete, or a path, or a cycle, then $c(G) = 2$.

b) If S has order at least 2, then every vertex u in S lies on a path uvw of order 3 such that $v \in V(G) \setminus H_G(S \setminus \{u\})$ *and* $w \in H_G(S \setminus \{u\})$.

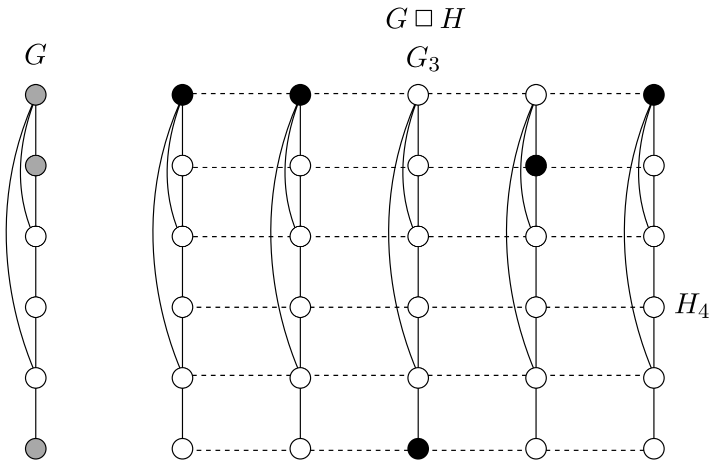


Figure 3: The projection onto G.

c) No proper subset S' of S satisfies $H_G(S') = V(G)$.

d) The convex hull $H_G(S)$ of S induces a connected subgraph of G.

As mentioned before, in [Barbosa et al., 2012] was proved that it is NP-hard to determine the Carathéodory number of the P_3 convexity of bipartite graphs. Observe that if G is a bipartite graph then $G \,\square\, K_2$ is also a bipartite graph. Therefore, as an immediate consequence of the result in [Barbosa et al., 2012] we can state the following result:

Corollary 2. *It is NP-hard to determine the Carathéodory number of the P_3 convexity on Cartesian product of graphs.*

In this section we denote $V(K_m) = V(P_m) = [m]$ and $E(P_m) = \{\{(i-1), i\} : 2 \leq i \leq n\}$. Our next result is on the Cartesian product of two complete graphs.

Theorem 3. *Consider $n, m \geq 2$ and $G = K_n \,\square\, K_m$. Then $c(G) = 2$.*

Proof. Let $G = K_n \,\square\, K_m$. Consider $S \subseteq K_n^i$ ($S \subseteq K_m^j$), for some $i \in [n]$ ($j \in [m]$) with $|S| \geq 2$ a Carathéodory set of G. As $H_G(S) = K_n^i$ ($H_G(S) = K_m^j$), any vertex $v \in H_G(S) \setminus S$ satisfy $v \in H_G(\{x, y\})$, where $x, y \in S$. So, $|S| = 2$. Now, consider $S \subseteq V(G)$, such that $S \cap K_n^i \cap K_m^j \neq \emptyset$ e $S \cap K_n^k \cap K_m^\ell \neq \emptyset$, with $i \neq k$ and $j \neq \ell$. Note that, $H_G(S) = V(G)$ and any vertex $v \in H_G(S) \setminus S$ satisfy $v \in H_G(\{x, y\})$, where $x \in (K_n^i \cap K_m^j)$ and $y \in (K_n^k \cap K_m^\ell)$. Then the maximum cardinality of a Carathéodory set in G is 2. □

The following lemma states that in the Cartesian product of G, a general graph with order at least two, and a complete graph K_m, each K_m-layer contains at most two vertices in some Carathéodory set.

Lemma 4. *Let G be a graph of order $n \geq 2$ and consider $G \,\square\, K_m$, $m \geq 2$. If S is a Carathéodory set of $G \,\square\, K_m$, then $|S \cap K_m^i| \leq 2$, for all $i \in [n]$.*

Proof. It is straightforward from Proposition 1 a). □

In order to determine the Carathéodory number of $P_n \,\square\, K_m$, we first set a lower bound by showing a construction of a Carathéodory set in these graphs.

Proposition 5. *Let $n, m \geq 2$ and $G = P_n \,\square\, K_m$. Then $c(G) \geq \lceil \frac{2n}{3} \rceil$ if $n \equiv 2 \pmod 3$ and $c(G) \geq \lceil \frac{2n}{3} \rceil + 1$, otherwise.*

Proof. Let $G = P_n \square K_m$ and $X = \{q \in \mathbb{Z}^+ : q \equiv 0, 1 \pmod 3 \text{ and } q \leq n\}$. Consider $S = \{(i,1) : i \in X\} \cup \{(1,2)\}$. Note that S has the enunciated cardinality. We have to show that S is a Carathéodory set of G. First consider $n \equiv 0, 1 \pmod 3$. We will show that $(n,2) \in \partial H_G(S)$. It is clear to see that $(n,2) \notin H_G(S \setminus \{(1,j)\})$, $j = 1, 2$. Now, let $(H_G)^i = H_G(S \setminus \{(i,1)\})$ for each $i \in X$ with $i \geq 3$. Then,

$$(H_G)^i = \{(k,j) : k = 1, \ldots, \ell \text{ and } j = 1, \ldots, m\} \cup$$

$$\{(k,1) : k = (\ell+3), \ldots, n\},$$

where $\ell = i-2$, if $i \equiv 0 \pmod 3$ or $\ell = i-1$, otherwise. Hence, $(n,2) \notin (H_G)^i$, for all $i \in X$.

If $n \equiv 2 \pmod 3$, we can see that $(n-1,2) \in \partial H_G(S)$ using analogous arguments. $\square$

The next result states that there cannot be two consecutive H-layers having empty intersection with a Carathéodory set of $P_n \square H$.

Lemma 6. *Consider the graph $G = P_n \square H$ such that H has order $m \geq 2$ and S is a Carathéodory set of G. If there exist H^i and H^j with $j > i$, $H^i \cap S \neq \emptyset$, $H^j \cap S \neq \emptyset$, and each H^k with $i < k < j$ has empty intersection with S, then $j \leq i+2$.*

Proof. Suppose S a Carathéodory set of G. Suppose that there exist H^i and H^j with $j > i+2$ and $H^i \cap S \neq \emptyset$ and $H^j \cap S \neq \emptyset$ and each H^k with $i < k < j$ has empty intersection with S. By the construction of $P_n \square H$ each vertex in H^k, with $i < k < j$, has at most one neighbor in $H_G(S)$ and $H_G(S)$ induces a disconnected graph. Hence, by Proposition 1 a), S is not a Carathéodory set of G. $\square$

The next result establishes that at most one K_m-layer of $P_n \square K_m$ may contain two vertices of a Carathéodory set.

Lemma 7. *Let $G = P_n \square K_m$ and S a Carathéodory set of G. Then, there is at most a K_m-layer, say i, such that $|S \cap K_m^i| = 2$.*

Proof. Suppose, by contradiction, that there exist two K_m-layers, say K_m^i and K_m^j, such that $|S \cap K_m^i| = 2$ and $|S \cap K_m^j| = 2$. Without loss of generality assume that $i < j$. Hence $(K_m^i \cup K_m^j) \subseteq H_G(S)$. Suppose a vertex $v \notin S$ such that $v \in H_G(S)$. If $v \in K_m^k$ for some $k < i$, then $v \in H_G(S \setminus K_m^j)$ and we can conclude that $v \notin \partial H_G(S)$. At the same way, if $k > j$, then $v \in H_G(S \setminus K_m^i)$ and we can conclude that $v \notin \partial H_G(S)$. Then, we may assume $i \leq k \leq j$. By Lemma 6, there not exist two consecutive K_m-layers between i and j with empty intersection with S. For every vertex $v \in K_m^k$, $v \in H_G(S \setminus \{x\})$, for any $x \in (S \cap (k_m^i \cup K_m^j))$. Therefore, $\partial H_G(S) = \emptyset$ and S is not a Carathéodory set of $P_n \square K_m$. $\square$

If S is a Carathéodory set of $P_n \square K_m$, then there are no three consecutive K_m-layers that contain some vertex in S.

Lemma 8. *Let $n, m \geq 2$, $G = P_n \square K_m$ and S a Carathéodory set of G. Let S' be the projection of S onto P_n. Then for all $i \in S'$, $|N_{P_n}[i] \cap S'| \leq 2$.*

Proof. Suppose, by contradiction, that $i \in S'$ such that $|N_{P_n}[i] \cap S'| = 3$ and $N_{P_n}(i) = \{i-1, i+1\}$. If $v \in (K_m^q \cap H_G(S))$ with $1 \leq q \leq i$, then $v \in H_G(S \setminus K_m^{i+1})$. Analogously if $v \in (K_m^q \cap H_G(S))$ with $i \leq q \leq n$, then $v \in H_G(S \setminus K_m^{i-1})$. Hence, S is not a Carathéodory set of G. $\square$

Using Lemmas 4, 7, and 8 we can establish an upper bound for the Carathéodory number of $P_n \square K_m$ and together with Proposition 5 we can determine its Carathéodory number.

Theorem 9. *Let $n, m \geq 2$ and $G = P_n \square K_m$. Then $c(G) = \lceil \frac{2n}{3} \rceil$ if $n \equiv 2 \pmod 3$ and $c(G) = \lceil \frac{2n}{3} \rceil + 1$, otherwise.*

Proof. Let S be a Carathéodory set of G with maximum cardinality. By Lemma 4, for any $i \in [n]$, $|K_m^i \cap S| \leq 2$. By Lemma 7, there is at most one K_m-layer with two vertices in S. By Lemma 8 there are no three consecutive K_m-layers that contain some vertex in S. Combining these previews results we have $|S| \leq \lceil \frac{2n}{3} \rceil$ if $n \equiv 2 \pmod 3$ and $|S| \leq \lceil \frac{2n}{3} \rceil + 1$, otherwise. Together with Proposition 5, we can conclude the proof of the statement. □

In Cartesian products of complete graphs and paths the Carathéodory number grows as the path grows. Differently, in Cartesian products of star graphs $K_{1,n}$ with complete graphs we have a fixed Carathéodory number, independently of the size of n. Since $K_{1,2}$ is isomorphic to P_3, we consider $n \geq 3$.

Proposition 10. *Let $n \geq 3$, $m \geq 2$ and $G = K_{1,n} \square K_m$. Then $c(G) = 3$.*

Proof. Let $V(K_{1,n}) = \{r\} \cup \{1, \ldots, n\}$, where the universal vertex is labeled by r, and $S = \{(r,1), (1,1), (2,2)\}$. It is easy to see that S is a Carathéodory set of G with $\partial H_G(S) = \{(1,2), \ldots, (1,m)\}$. Now, we will show that there is not a Carathéodory set of G of size 4 in G.

Remember that, by Lemma 4, $|S \cap K_m^i| \leq 2$, for all $i \in ([n] \cup \{r\})$. Note that, if $K_m^i \subseteq H_G(S)$, for some $i \in [n]$, then $1 \leq |K_m^i \cap S| \leq 2$. Furthermore, if $|K_m^i \cap S| = 1$, $K_m^r \subseteq H_G(S)$. We have three cases related to the number of the vertices of K_m^r in S.

Case 1: $|K_m^r \cap S| = 2$.

If $K_m^r \cap S = \{(r,a), (r,b)\}$ for some $a, b \in [m]$ with $a \neq b$, then $K_m^r \subseteq H_G(S)$. Consider the sets $A = \{(c,k) : c \in [n] \text{ and } k \in \{a,b\}\}$ and $B = \{(x,y) : x \in [n] \text{ and } y \in ([m] \setminus \{a,b\})\}$. So, $A \cap S = \emptyset$, in view to avoid an induced P_3 by the vertices in S (by Proposition 1 c)). Any vertex $(i,j) \in A \cup B$ has a neighbor in K_m^r and its other neighbors are in the same K_m-layer. Thus, if $(i,j) \in H_G(S) \setminus S$, it has a neighbor (i,d) in $K_m^i \cap S$, for some $d \in ([m] \setminus \{a,b\})$. Hence $(i,j) \in H_G(\{(r,a), (r,b), (i,d)\})$ and there is no vertex of $A \cup B$ that needs more than 3 vertices in S to be in $H_G(S)$. Thus, $|B \cap S| \leq 1$. See Figure 4(a) for an illustration.

Case 2: $|K_m^r \cap S| = 1$.

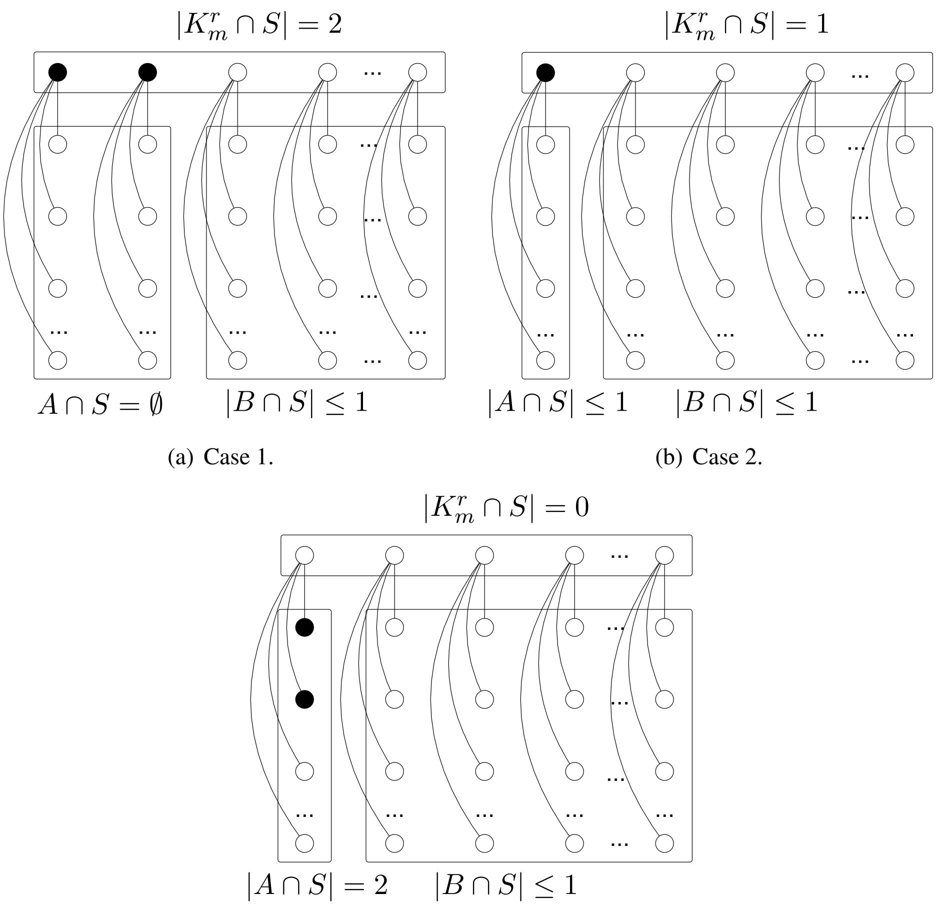
Let $K_m^r \cap S = \{(r,a)\}$ for some $a \in [m]$. Now, consider the sets $A = \{(c,a) : c \in [n]\}$ and $B = \{(x,y) : x \in [n] \text{ and } y \in ([m] \setminus \{a\})\}$. So, $|A \cap S| \leq 1$, in view to avoid an induced P_3 in S. If $A \cap S = \{(i,a)\}$, $K_m^i \cap S = \{(i,a)\}$, i.e., there is not other vertex in the same K_m-layer of (i,a) in S. If $(i,b) \in (B \cap S)$, then $K_m^r \subseteq H_G(S)$ and, similar to Case 1, at most a vertex of B belongs to S. Thus, $c(G) \leq 3$. See Figure 4(b) for an illustration.

Case 3: $|K_m^r \cap S| = 0$.

First suppose that there is a $K_{1,n}$-layer, say a, such that $|K_{1,n}^a \cap S| \geq 3$. If $S \subseteq K_{1,n}^a$, then $H_G(S) = S$ and S is not a Carathéodory set of G. Then suppose that some $K_{1,n}^b \cap S \neq \emptyset$, with $a \neq b$. So, $K_m^r \subseteq H_G(S)$. But every vertex in $K_m^r \cap (H_G(S) \setminus S)$ belongs to $H_G(S)$ for some S with at most three vertices. Any other vertex in $(H_G(S) \setminus S)$ must have a neighbor in S in the same K_m-layer and another in K_m^r, then it also belongs to $H_G(S)$ for some S with at most three vertices. If every $K_{1,n}$-layer has at most a vertex in S, $K_m^r \notin H_G(S)$ and S is not a Carathéodory set since every vertex in $H_G(S) \setminus S$ needs only vertices of the same K_m-layer to belong to $H_G(S)$. So, we may assume that $|K_{1,n}^a \cap S| = 2$, for some $a \in [m]$. Consider the set $B = \{(x,y) : x \in [n]$ and $y \in ([m] \setminus \{a\})\}$. Again, with the same argument of Case 1, we can conclude that at most a vertex of B belongs to S. Thus, $c(G) \leq 3$. See Figure 4(c) for an illustration.

Thus, we can conclude $c(G) = 3$.

□



(a) Case 1. (b) Case 2. (c) Case 3.

Figure 4: An illustration of proof of Proposition 10. For the sake of simplicity, all edges of K_m-layers are omitted.

A *full binary tree* is a binary tree of height h that contains exactly $2^{h+1} - 1$ vertices. For a full binary tree T_h with root r and height $h \geq 1$ we denote

$$V(T_h) = \{r\} \cup \{2^i, (2^i + 1) \ldots, (2^{i+1} - 1), \ldots, 2^h, (2^h + 1), \ldots, (2^{h+1} - 1)\},$$

where i is the distance from r to the respective vertice. With this notation, vertices with label $2^h, (2^h + 1), \ldots, (2^{h+1} - 1)$ are the leaves of T_h.

In our last result, we show how to construct a Carathéodory set in the Cartesin product of a full binary tree and a complete graph from Carathéodory sets of smaller trees. In [Barbosa et al., 2012], the authors shows that binary (sub)trees play a central role for the Carathéodory number of P_3 convexity.

Theorem 11. *Let T_h be a full binary tree with root r and height $h \geq 1$. Let $G = T_h \,\square\, K_m$, with $m \geq 3$. Then there is a Carathéodory set S of G such that*

1. $H_G(S) = V(G)$*,*

2. $K_m^r \subseteq \partial H_G(S)$*, and*

3. *if* $h = 1$*, then* $|S| = 3$ *and* $|S| = 3(2^{h-1})$*, otherwise.*

Proof. We prove the statement by induction on the height h of T. If $h = 1$, then it is easy to see that the set $\{(2, 1), (2, 2), (3, 3)\}$ is a Carathéodory set of G with $H_G(S) = V(G)$ and $K_m^r \subseteq \partial H_G(S)$. Hence let $h \geq 2$. Let r_1 and r_2 be the two children of r in T_h. For $i \in \{1, 2\}$, let $(T_{h-1})^i$ be the full binary subtree of T_h containing r_i and all descendants of r_i. By induction there is a Carathéodory set S_i of $G_i = (T_{h-1})^i \,\square\, K_m$ such that $H_{G_i}(S) = V(G_i)$, $K_m^{r_i} \subseteq \partial H_G(S_i)$ and $|S_i| = 3$ if $h - 1 = 1$ and $|S| = 3(2^{h-2})$, otherwise. Now, let $S = S_1 \cup S_2$. We have $|S| = 2(3(2^{h-2}))$, which is equal $3(2^{h-1})$. Since $T_h \,\square\, K_m$ is the graph induced by $V((T_{h-1})^i) \cup K_m^r$, and r has exactly the two neighbours r_1 and r_2 in T_h, this implies that in $G = T_h \,\square\, K_m$, every vertex of K_m^r has exactly one neighbor in each $H_{G_i}(S_i)$ and then $H_G(S) = V(G_i) \cup K_m^r$. Since $K_m^{r_i} \subseteq \partial H_{G_i}$, $K_m^r \subseteq \partial H_G(S)$ and the proof is complete. See in Figure 5 a Carathéodory set of $G = T_2 \,\square\, K_3$. □

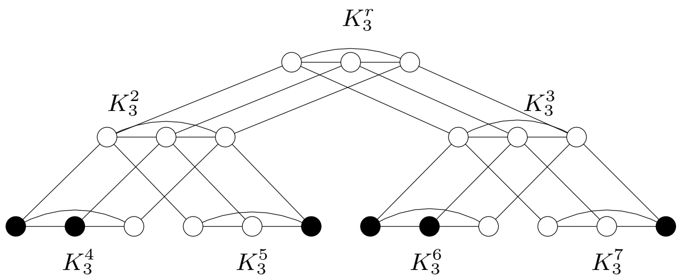


Figure 5: Graph G, that is a Cartesian product of a full binary tree T_2 of height $h = 2$ with a complete graph K_3. The black vertices are in a Carathéodory set S of G with cardinality $3(2^{h-1}) = 6$ and K_3^r is a subset of $\partial H_G(S)$.

4. Final considerations

In this work we establish the Carathéodory number for some Cartesian products in the P_3 convexity. The Cartesian product is well studied for other problems in graphs but there are few results on the Carathéodory number. Motivated by this we determine the Carathéodory number for the Cartesian product of K_n, P_n and $K_{1,n}$ with K_m. Also, we present a recursive way to construct a Carathéodory set in $T_h \,\square\, K_m$, where T_h is a full binary tree with height h. Some suggestions for future work are studying the Carathéodory number in the P_3 convexity for the Cartesian product of $P_n \,\square\, P_m$, $G \,\square\, P_m$ and $G \,\square\, K_m$ and establish limits for Cartesian product of $G \,\square\, H$ for general graphs G and H.

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